NAME: ______________________________

SECTION: (Epstein 11:15) [ ]

(Groszek 1:45) [ ]

Math 8

Winter 2015

Exam I

Instructions:

1. Write your name *legibly* on this page, and indicate your section by checking the appropriate box.

2. There are five problems, some of which have multiple parts. Do all of them.

3. Explain what you are doing, and show your work. You will be *graded on your work*, not just on your answer. Make it clear and legible so we can follow it.

4. It is fine to leave your answer in a form such as $\ln(.02)$ or $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $e^{\ln(.02)}$ or $\cos(\pi)$ or $(3-2)$), you should simplify it.

5. There are a few pages of scratch paper at the end of the exam. We *will not look* at these pages unless you write on a problem "Continued on page..."

6. This exam is closed book. You may not use notes, calculators, or any other external resource. It is a violation of the honor code to give or receive help on this exam.

Problem	Points	Score
1	25	
2	20	
3	10	
4	20	
5	25	
Total	100	

1. (25 points.)

(a) (15 points.) Use a trigonometric substitution to rewrite the following definite integral in the form $\int_a^b f(\theta)\,d\theta$, where the expression $f(\theta)$ does not contain any radical signs. Do NOT evaluate the integral, just rewrite it using the substitution.

$$\int_4^6 x^2\sqrt{x^2-4x}\,dx$$

Solution: Complete the square to rewrite the expression under the radical sign: $x^2-4x=(x-2)^2-4$. Use the trigonometric identity $\sec^2\theta-1=\tan^2\theta$ in the form $4\sec^2\theta-4=4\tan^2\theta$, and use the substitution $(x-2)^2=4\sec^2\theta$, or

$$x=2\sec\theta+2.$$

This gives us

$$dx=2\sec\theta\tan\theta\,d\theta,$$

and

$$x^2-4x=(x-2)^2-4=4\sec^2\theta-4=4\tan^2\theta.$$

The bounds of integration are

$$x=4 \qquad 2\sec\theta+2=4 \qquad \sec\theta=1 \qquad \theta=\sec^{-1}(1)=0;$$

$$x=6 \qquad 2\sec\theta+2=6 \qquad \sec\theta=2 \qquad \theta=\sec^{-1}(2)=\frac{\pi}{3};$$

The integral now becomes

$$\int_4^2 x^2\sqrt{x^2-4x}\,dx=\int_0^{\frac{\pi}{3}}(2\sec\theta+2)^2\sqrt{4\tan^2\theta}\sec\theta\tan\theta\,d\theta=$$

$$\int_0^{\frac{\pi}{3}}4(2\sec\theta+2)^2|\tan\theta|\sec\theta\tan\theta\,d\theta$$

Since $0\le\theta\le\frac{\pi}{2}$, we know that $\tan\theta$ is positive, and so $|\tan\theta|=\tan\theta$. Therefore, our final answer is

$$\int_0^{\frac{\pi}{3}}4(2\sec\theta+2)^2\tan^2\theta\sec\theta\,d\theta$$

(b) (10 points.) Use a u-substitution to rewrite the following definite integral in the form $\int_a^b f(u)\,du$, where the expression $f(u)$ does not contain any trigonometric functions. Do NOT evaluate the integral, just rewrite it using the substitution.

$$\int_0^{\frac{\pi}{4}} \tan^4 x \sec^{-2} x\, dx$$

Solution: Rewrite the integral as

$$\int_0^{\frac{\pi}{4}} \frac{\tan^4 x}{\sec^2 x}\, dx = \int_0^{\frac{\pi}{4}} \frac{\tan^4 x}{(\sec^2 x)^2} \sec^2 x\, dx$$

and make a substitution

$$u = \tan x \qquad du = \sec^2 x\, dx \qquad \sec^2 x = \tan^2 x + 1 = u^2 + 1$$

$$0 \le x \le \frac{\pi}{4} \qquad 0 \le \tan x \le 1 \qquad 0 \le u \le 1$$

$$\int_0^{\frac{\pi}{4}} \tan^4 x \sec^{-2} x\, dx = \int_0^1 \frac{u^4}{(u^2+1)^2}\, du.$$

2. (20 points.) Evaluate the following. (Your answer should be a number, $+\infty$, $-\infty$, or "diverges" if it diverges but not to $+\infty$ or $-\infty$.) Be sure to show your answer is correct.

(a) $\lim_{n\to\infty} a_n$ where

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$$

Solution: If we set $b_n = 0$ and $c_n = \dfrac{1}{n}$, then we have $b_n \leq a_n \leq c_n$ for all n. Since $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$, by the squeeze theorem for sequences, $\lim_{n\to\infty} a_n = 0$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{5(-3)^n}{5^n}$

Solution: This is a geometric series, with first term $a = -3$ and ratio between terms $r = -\frac{3}{5}$. Since $|r| < 1$, the series converges to $\dfrac{a}{1-r} = \dfrac{-3}{\frac{8}{5}} = \dfrac{-15}{8}$.

3. (10 points.) You are given $a_n = \dfrac{(-1)^n}{(n+3)^3}$.

Assume you want to use the partial sum $s_c = \displaystyle\sum_{n=1}^{c} a_n$ to approximate the value of $\displaystyle\sum_{n=1}^{\infty} a_n$.

To guarantee an error of at most $\dfrac{8}{10^6}$, how large must c be?

Solution: Since $\dfrac{1}{(n+3)^3}$ is positive and decreasing for $n \geq 0$, this is an alternating series, and we can use the error estimate from the alternating series test:

$$|R_c| \leq |a_{c+1}| = \frac{1}{(c+4)^3}$$

We want

$$\frac{1}{(c+4)^3} \leq \frac{8}{10^6} \qquad (c+4)^3 \geq \frac{10^6}{8} \qquad (c+4) \geq \frac{10^2}{2} \qquad c \geq 46.$$

4. (20 points.) Find the volume of the solid obtained by revolving the area under the curve $y = \ln x$, for $1 \le x \le e$, around the x-axis.

Solution: The cross-sectional area at x is the area of a circle with radius $\ln x$, or $A(x) = \pi(\ln x)^2$. Therefore the volume is

$$\int_1^e \pi(\ln x)^2\,dx = \pi \int_1^e (\ln x)^2\,dx.$$

Evaluate the indefinite integral using integration by parts:

$$u = (\ln x)^2 \qquad dv = dx \qquad du = \frac{2\ln x}{x}\,dx \qquad v = x.$$

(Since x is positive on the interval we are integrating over, we can use $\frac{1}{x}$ in place of $\frac{1}{|x|}$ as the derivative of $\ln x$.) We get

$$\int u\,dv = uv - \int v\,du$$

$$\int (\ln x)^2\,dx = x(\ln x)^2 - \int x\frac{2\ln x}{x}\,dx = x(\ln x)^2 - 2\int \ln x\,dx =$$
$$x(\ln x)^2 - 2(x\ln x - x) + C = x(\ln x)^2 - 2x\ln x + 2x + C.$$

(If you don't remember the antiderivative of $\ln x$, you can use integration by parts again to find it.) Now

$$\pi\int_1^e (\ln x)^2\,dx = \pi\Big[x(\ln x)^2 - 2x\ln x + 2x\Big]\Big|_{x=1}^{x=e} = \pi((e - 2e + 2e) - (2)) = \pi(e-2).$$

5. (25 points.) Determine whether the following converge or diverge. Be sure to show your answer is correct.

   (a) (10 points.) $\sum_{n=1}^{\infty} 5e^{-n}$.

   Solution: Since $f(x) = 5e^{-x}$ is positive, continuous, and decreasing, we can use the integral test.

$$\int_1^\infty 5e^{-x}\,dx = \lim_{b\to\infty} \int_1^b 5e^{-x}\,dx =$$

$$= \lim_{b\to\infty} \Big[-5e^{-x} \Big]\Bigg|_{x=1}^{x=b} = \lim_{b\to\infty} \Big[-5e^{-b} + 5e^{-1} \Big] = 5e^{-1}.$$

   Since the integral converges, the series also converges.

(b) (15 points.) $\displaystyle\sum_{n=1}^{\infty} \frac{n+\sqrt{n}}{n^2+7}$.

Solution: This series appears to diverge, because for large values of n, the terms seem to be close to the terms of the harmonic series, $\dfrac{1}{n}$, which we know diverges. We first use the comparison test. Since $\dfrac{n+\sqrt{n}}{n^2+7} \geq \dfrac{n}{n^2+7}$, if we can show the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+7}$ diverges, then we will know our series diverges.
This series looks very much like the harmonic series. We can compare them using the limit comparison test.

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n}{n^2+7}} = \lim_{n\to\infty} \frac{n^2+7}{n^2} = 1.$$

Since this limit is neither 0 nor ∞, and the harmonic series diverges, by the limit comparison test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+7}$ diverges also. Therefore, by the comparison test, our original series diverges.

Note: As usual, there are other ways to do this problem.

For example, we could compare our original series to the series $\sum_{n=1}^{\infty} \frac{n}{n^2 + n^2} = \sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{n}$, which must diverge because the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Or, we could start in the same way by comparing our series to $\sum_{n=1}^{\infty} \frac{n}{n^2+7}$, and then show that the series $\sum_{n=1}^{\infty} \frac{n}{n^2+7}$ diverges by using the integral test. (The integral is not hard to compute with a u-substitution.)

Or, we could use the limit comparison test right away, to compare our original series to the harmonic series.

Extra paper for scratch work. We will not look at this page unless you write a note on a problem saying "Continued on page…"

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