NAME : ______________________________

SECTION : (circle one) 12:30–1:35 1:45–2:50

Math 8

March 12, 2010

Final Exam

SOLUTIONS

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem. You have three hours and you should attempt all problems.

- Calculators or other computing devices are not allowed.
- Use the blank page at the end of the exam for scratch work.
- Unless a problem indicates otherwise, you must show all work and give a reason (or reasons) for your answer. A correct answer with incorrect work will be considered wrong.

1. (20) Compute the Taylor polynomial of degree 2 centered at $x = 0$ for the function

$$f(x) = \sqrt{1+x}$$

$f(x) = (1+x)^{1/2}$	$f(0) = 1$
$f'(x) = \frac{1}{2}(1+x)^{-1/2}$	$f'(0) = \frac{1}{2}$
$f''(x) = \frac{-1}{4}(1+x)^{-3/2}$	$f''(0) = -\frac{1}{4}$

$$T_2(x) = 1 + \frac{1/2}{1!}x + \frac{-1/4}{2!}x^2 = \boxed{1 + \frac{1}{2}x - \frac{1}{8}x^2.}$$

(b) Use the Remainder Theorem to give a bound on the error involved in using this Taylor polynomial to approximate $f(x)$ at $x = 1$.

$R_2(x) = \dfrac{f'''(c)}{3!}(x-0)^3$ for some c between 0 and 1.

$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$. Since this is a decreasing function, $f'''(c) \leq \frac{3}{8}$ for all c between 0 and 1. Therefore:

$$\text{Error} \leq \frac{3/8}{3!}(1)^3 = \boxed{\frac{1}{16}.}$$

2. (20) Determine whether the following series are conditionally convergent, absolutely convergent, or divergent. Mention any test that you might use and verify that it is applicable.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Integral test:

$$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b \to \infty} \int_2^b \frac{dx}{x \ln x}$$

Set $u = \ln x$,

$du = \frac{1}{x} dx$,

$$= \lim_{b \to \infty} \int_{x=2}^{x=b} \frac{du}{u} = \lim_{b \to \infty} \ln u \Big|_{x=2}^{x=b}$$

$$= \lim_{b \to \infty} \ln \ln b - \ln \ln 2 = \infty.$$

This series $\boxed{\text{diverges}}$ by the Integral test.

(b) $\sum_{n=1}^{\infty} \frac{\sin^2(n)\cos^3(n)}{n^3+2n}$

Test for absolute convergence:

$$\sum_{n=1}^{\infty} \left| \frac{\sin^2(n)\cos^3(n)}{n^3+2n} \right|.$$

Since

$$\left| \frac{\sin^2(n)\cos^3(n)}{n^3+2n} \right| \leq \frac{1}{n^3},$$

the given series $\boxed{\text{converges absolutely}}$ by comparison to a p-series.

3. (20) Determine the interval of convergence of the power series

$$\sum_{n=3}^{\infty} \frac{(3x-2)^n}{n^{3/2}}.$$

Use the Ratio test to find the interval:

$$\text{Ratio} = \left| \frac{\frac{(3x-2)^{n+1}}{(n+1)^{3/2}}}{\frac{(3x-2)^n}{n^{3/2}}} \right| = \left| (3x-2) \frac{n^{3/2}}{(n+1)^{3/2}} \right|$$

$$\rightarrow |3x-2|.$$

So we need $|3x-2| < 1$

$$-1 < 3x-2 < 1$$

$$1 < 3x < 3$$

$$\tfrac{1}{3} < x < 1.$$

At $\frac{1}{3}$: $\sum \frac{(-1)^n}{n^{3/2}}$ converges (absolutely, even).

At 1: $\sum \frac{1}{n^{3/2}}$ converges (p-series).

Interval of convegence: $\boxed{[\tfrac{1}{3}, 1].}$

4. (20) Evaluate

$$\lim_{x\to 0} \frac{\cos x^2 - 1 + x^4/2}{x^8}$$

using Taylor series.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots$$

$$\cos x^2 - 1 + \frac{x^4}{2} = \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots$$

Therefore,

$$\lim_{x\to 0} \frac{\cos x^2 - 1 + \frac{x^4}{2}}{x^4}$$

$$= \lim_{x\to 0} \frac{1}{4!} - \frac{x^4}{6!} + \cdots$$

$$= \boxed{\frac{1}{4!}}$$

5. (20) Let

$$\mathbf{r}(t) = \left\langle t^2 + 5, \frac{4t^{3/2}}{3}, t - 6 \right\rangle$$

be a curve in 3-space from $t = 0$ to $t = 1$. Find the length of the curve.

$$\text{Speed} = |\vec{v}(t)| = |\langle 2t, 2\sqrt{t}, 1 \rangle|$$

$$= \sqrt{4t^2 + 4t + 1}$$

$$= 2t + 1.$$

$$\text{Arc length} = \int_0^1 \text{Speed } dt$$

$$= \int_0^1 2t + 1 \, dt$$

$$= t^2 + t \Big|_0^1$$

$$= \boxed{2.}$$

6. (20) Find the equation of the tangent plane to the surface defined by

$$xyz + \sqrt{3x + yz} = 3$$

at the point $(1, 1, 1)$.

Set $F = xyz + \sqrt{3x+yz}$.

$$\nabla F = \left\langle yz + \frac{3}{2}(3x+yz)^{-1/2},\ xz + \frac{z}{2}(3x+yz)^{-1/2},\ yz + \frac{y}{2}(3x+yz)^{-1/2} \right\rangle.$$

$$\nabla F(1,1,1) = \left\langle 1 + \frac{3}{4},\ 1 + \frac{1}{4},\ 1 + \frac{1}{4} \right\rangle$$

$$= \left\langle \frac{7}{4},\ \frac{5}{4},\ \frac{5}{4} \right\rangle.$$

Tangent plane:

$$\nabla F \cdot (\vec{r} - \vec{r}_0) = 0,$$

$$\boxed{\frac{7}{4}(x-1) + \frac{5}{4}(y-1) + \frac{5}{4}(z-1) = 0.}$$

7. (20) Find and classify all critical points of the function

$$f(x, y) = x^4 + y^4 - 4xy + 1.$$

$f_x = x^3 - 4y \Rightarrow y = x^3$

$f_y = 4y^3 - 4x \Rightarrow 4x^9 = 4x \Rightarrow x = 0, \pm 1$

Critical points are $(0,0)$, $(-1,-1)$, $(1,1)$.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix}$$

$$= 144x^2y^2 - 16.$$

$D(0,0) = -16 < 0$, so $\boxed{(0,0) \text{ is a saddle.}}$

$D(1,1) = 144 > 0$, and $f_{xx}(1,1) = 12 > 0$, so $\boxed{(1,1) \text{ is a local min.}}$

$D(-1,-1) = 144 > 0$, and $f_{xx}(-1,-1) = 12 > 0$, so $\boxed{(-1,-1) \text{ is a local min.}}$

8. (10) Evaluate the integral

$$\int \ln x \, dx$$

using integration by parts.

$u = \ln x \qquad v = x$

$du = \frac{1}{x} dx \qquad dv = dx$

$$\int \ln x \, dx = x \ln x - \int \frac{x}{x} dx$$

$$= \boxed{x \ln x - x + C}$$

9. (10) Evaluate the integral

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$$

using a trigonometric substitution. Your final answer should not contain any trigonometric functions.

Since $4\tan^2\theta + 4 = 4\sec^2\theta$, we substitute

$$x = 2\tan\theta$$
$$dx = 2\sec^2\theta\,d\theta$$

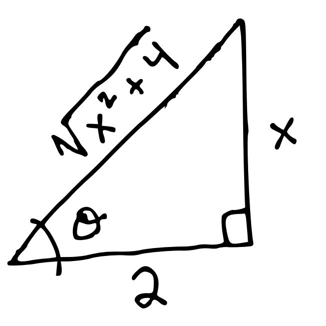
$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{2\sec^2\theta}{(4\tan^2\theta)(2\sec\theta)}\,d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

Set $u = \sin\theta$, $du = \cos\theta\,d\theta$:

$$= \frac{1}{4}\int u^{-2}\,du = \frac{-1}{4}\frac{1}{u} + C = \frac{-1}{4\sin\theta} + C.$$

Now return to x's:



$\sin\theta = \dfrac{x}{\sqrt{x^2+4}}$, so

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \boxed{\frac{-\sqrt{x^2+4}}{4x} + C}$$

10. (10) Match the functions A–E with their Taylor series a–e. You need not show any work.

A	$\sin(2x)$
B	$\cos(2x)$
C	$x\cos(2x)$
D	e^{-4x}
E	$\int_0^{2x} e^{-t^2}\,dt$

(a) $\sum_{n=0}^{\infty}(-4)^n \frac{x^{2n+1}}{(2n)!}$. C

(b) $\sum_{n=0}^{\infty}(-4)^n \frac{x^{2n}}{(2n)!}$. B

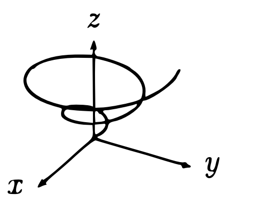
(c) $\sum_{n=0}^{\infty}2(-4)^n \frac{x^{2n+1}}{(2n+1)n!}$. E

(d) $\sum_{n=0}^{\infty}2(-4)^n \frac{x^{2n+1}}{(2n+1)!}$. A

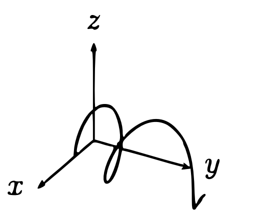
(e) $\sum_{n=0}^{\infty}(-4)^n \frac{x^n}{n!}$. D

11. (10) Match the vector functions A–E with their graphs a–e. You need not show any work.

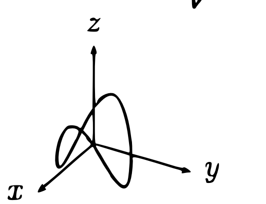
A	$\langle \cos t, \sin t \cos 2t \rangle$
B	$\langle \cos t, \sin t, 2 \rangle$
C	$\langle t, t^2, t \rangle$
D	$\langle \cos t, t^2, \sin t \rangle$
E	$\langle t \cos t, t \sin t, t \rangle$

(a) 

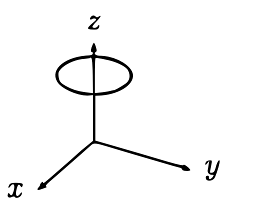
E

(b) 

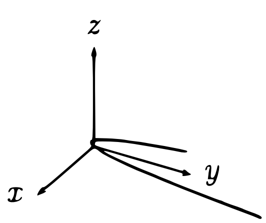
D

(c) 

A

(d) 

B

(e) 

C

12. (5) Suppose that $\sum_{n=1}^{\infty} a_n = 3$. What is $\lim_{n\to\infty} a_n$? You need not show any work.

$\boxed{\lim a_n = 0.}$

(otherwise the series would diverge.)

13. (5) List all third order partial derivatives of the function

$$f(x, y) = x^3 - 2xy^2.$$

$f_x = 3x^2 - 2y^2$

$f_y = -4xy$

$f_{xx} = 6x$

$f_{xy} = -4y$

$f_{yy} = -4$

$$\boxed{\begin{aligned} f_{xxx} &= 6 \\ f_{xxy} &= 0 \\ f_{xyy} &= -4 \\ f_{yyy} &= 0 \end{aligned}}$$

14. (5) What is the maximum value of a directional derivative of the function

$$f(x, y) = \frac{1}{1 + x^2 + y^2}$$

at the point $(1, 1, 1/3)$?

$$\nabla f = \left\langle \frac{-2x}{1+x^2+y^2}, \frac{-2y}{1+x^2+y^2} \right\rangle$$

$$\nabla f(1,1) = \left\langle \frac{-2}{9}, \frac{-2}{9} \right\rangle$$

Maximum value of a directional derivative:

$$|\nabla f(1,1)| = \sqrt{\frac{4}{81} + \frac{4}{81}} = \boxed{\sqrt{\frac{8}{81}}}$$

15. (5) The function $f(x, y) = x^2 + y^4$ has a critical point at the origin. Classify this critical point as a local minimum, a local maximum, or a saddle point. You need not show any work.

$f(0,0) = 0$.

Since $f(x,y)$ cannot be negative, $(0,0)$ must be a $\boxed{\text{local minimum.}}$

Note that the Second Derivative Test is inconclusive in this case.