NAME : ______________________________

SECTION : (circle one) Arkowitz Vatter

Math 8

March 16, 2009

Final Exam

$\boxed{\text{SOLUTIONS}}$

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem. You have three hours and you should attempt all problems.

- ***Print*** your name in the space provided and circle your instructor's name.
- Calculators or other computing devices are not allowed.
- Use the blank page at the end of the exam for scratch work.
- **Except in the multiple choice section, you must show all work and give a reason (or reasons) for your answer. A CORRECT ANSWER WITH INCORRECT WORK WILL BE CONSIDERED WRONG.**

1. (10) Express the integral

$$\int \frac{\sin x - x}{x}\,dx$$

as an infinite series. Your answer should be in the form $\sum_{n=0}^{\infty} c_n x^n$, for some coefficients c_n. (In other words, do *not* simply give the first few terms.)

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin x - x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\frac{\sin x - x}{x} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

$$\int \frac{\sin x - x}{x}\,dx = \boxed{\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} + C}$$

2. (16) Determine the integral

$$\int \frac{\sqrt{x^2-1}}{x^3}\,dx.$$

$\tan^2\theta = \sec^2\theta - 1$, so set $x = \sec\theta$.

$dx = \sec\theta\tan\theta\,d\theta$.

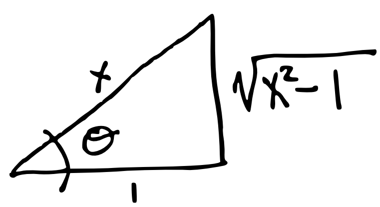
$$\int \frac{\sqrt{x^2-1}}{x^3}\,dx = \int \frac{\sqrt{\sec^2\theta - 1}}{\sec^3\theta}\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \int \sin^2\theta\,d\theta$$

$$= \int \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$



$$\sin 2\theta = 2\sin\theta\cos\theta = 2\,\frac{\sqrt{x^2-1}}{x}\,\frac{1}{x},\ \text{so}$$

$$\int \frac{\sqrt{x^2-1}}{x^3}\,dx = \boxed{\frac{1}{2}\operatorname{arcsec} x - \frac{\sqrt{x^2-1}}{2x^2} + C}$$

3. (14) Determine the integral

$$\int \frac{x^2}{2} f''(x)\, dx.$$

Your answer should contain $f(x)$, $f'(x)$ and $\int f(x)\, dx$.

Integration by parts (twice):

$$u = \frac{x^2}{2} \qquad v = f'(x)$$

$$du = x\, dx \qquad dv = f''(x)\, dx$$

$$\int \frac{x^2}{2} f''(x)\, dx = \frac{x^2}{2} f'(x) - \int x\, f'(x)\, dx$$

$$u = x \qquad v = f(x)$$

$$du = dx \qquad dv = f'(x)\, dx$$

$$= \boxed{\frac{x^2}{2} f'(x) - x f(x) + \int f(x)\, dx.}$$

4. (16) A particle moves in space and its acceleration at time t is given by

$$\vec{a}(t) = \langle e^t, 0, e^{-t} \rangle.$$

Its initial velocity is $\vec{v}(0) = \langle 1, \sqrt{2}, -1 \rangle$ and its inital position is $\vec{s}(0) = \langle 3, 0, 2 \rangle$.

(a) What is the particle's velocity and position at any time t?

$$\vec{v}(t) = \int \vec{a}(t)\, dt = \langle e^t, 0, -e^{-t} \rangle + \vec{C}$$

$$\vec{v}(0) = \langle 1, \sqrt{2}, -1 \rangle = \langle 1, 0, -1 \rangle + \vec{C}, \text{ so}$$

$$\boxed{\vec{v}(t) = \langle e^t, \sqrt{2}, -e^{-t} \rangle}$$

$$\vec{s}(t) = \int \vec{v}(t)\, dt = \langle e^t, \sqrt{2}t, e^{-t} \rangle + \vec{C}$$

$$\vec{s}(0) = \langle 3, 0, 2 \rangle = \langle 1, 0, 1 \rangle + \vec{C}, \text{ so}$$

$$\boxed{\vec{s}(t) = \langle e^t + 2, \sqrt{2}t, e^{-t} + 1 \rangle}$$

(b) What is the distance the particle has moved (along its path) from time $t = 0$ to time $t = \ln 2$. Simplify your answer by expressing it as a decimal.

$$\text{Speed} = |\vec{v}(t)| = \sqrt{e^{2t} + 2 + e^{-2t}} = e^t + e^{-t}$$

$$\text{Distance travelled} = \int \text{Speed}$$

$$= \int_0^{\ln 2} e^t + e^{-t}\, dt$$

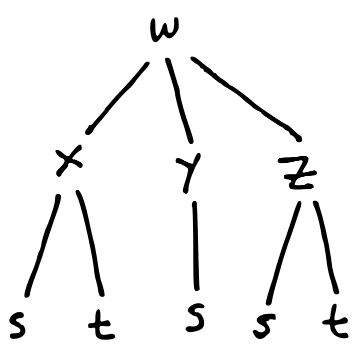
$$= e^t - e^{-t} \Big|_0^{\ln 2}$$

$$= 2 - 1 - \frac{1}{2} + 1$$

$$= \boxed{1.5}$$

5. (10) Let $w = xy + z$ and let $x = s\cos t$, $y = \ln s$ and $z = \arctan(s+t)$. Find

$$\frac{\partial w}{\partial s}.$$



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{dy}{ds} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial w}{\partial s} = (y)(\cos t) + (x)\left(\frac{1}{s}\right) + (1)\left(\frac{1}{1+(s+t)^2}\right)$$

$$= \boxed{(\ln s)(\cos t) + (s\cos t)\left(\frac{1}{s}\right) + (1)\left(\frac{1}{1+(s+t)^2}\right).}$$

6. (16) The function $z = f(x,y)$ has, at the point $(1,2)$, $D_{\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle}(1,2) = 7/\sqrt{2}$ and $D_{\langle 1/\sqrt{5}, 2/\sqrt{5}\rangle}(1,2) = 11/\sqrt{5}$. What is the directional derivative at $(1,2)$ in the direction from $(1,2)$ to $(3,5)$?

The direction from $(1,2)$ to $(3,5)$ is $\langle 2,3\rangle$.

As a unit vector, this is $\left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$.

$$D_{\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle} = \frac{7}{\sqrt{2}} = \frac{1}{\sqrt{2}}\frac{\partial f}{\partial x} + \frac{1}{\sqrt{2}}\frac{\partial f}{\partial y}$$

$$D_{\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle} = \frac{11}{\sqrt{5}} = \frac{1}{\sqrt{5}}\frac{\partial f}{\partial x} + \frac{2}{\sqrt{5}}\frac{\partial f}{\partial y}.$$

Solving this shows $\frac{\partial f}{\partial x} = 3$ and $\frac{\partial f}{\partial y} = 4$.

$$D_{\left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle} = \frac{6+12}{\sqrt{13}} = \boxed{\frac{18}{\sqrt{13}}}$$

7. (14) Find the equation of the tangent plane to the surface defined by

$$\underbrace{x - z = 4\arctan(yz)}$$

at $(1+\pi, 1, 1,)$.

$$\underbrace{4\arctan(yz) - x + z}_{F(x,y,z).} = 0$$

$$\nabla F = \left\langle -1, \frac{4z}{1+(yz)^2}, \frac{4y}{1+(yz)^2} + 1 \right\rangle$$

$$\nabla F(1+\pi, 1, 1) = \langle -1, 2, 3 \rangle.$$

Tangent plane ($\perp$ to ∇F):

$$\boxed{-1\left(x - (1+\pi)\right) + 2(y-1) + 3(z-1) = 0.}$$

8. (14) Determine the local maximum point(s), local minimum point(s) and saddle point(s) (if any of these exist) of the function

$$z = x^3 + y^3 - 9xy + 27.$$

You may use the second derivative test stated at the beginning of the exam.

Critical points:

$f_x = 3x^2 - 9y$

$f_y = 3y^2 - 9x$

Solving this gives critical points $(0,0)$, $(3,3)$.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -9 \\ -9 & 6y \end{vmatrix} = 36xy - 81.$$

$D(0,0) = -81 < 0$, so that is a $\boxed{\text{saddle.}}$

$D(3,3) = 1215 > 0$, and $f_{xx}(3,3) = 18 > 0$, so that is a $\boxed{\text{local min.}}$

1. (6) What is the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n n!}$?

Ratio Test:

A. $(-5,-1]$

B. $[-5,1)$

C. $(-1,5]$

D. $[-1,5)$

E. $(-\infty,\infty)$

$$\left| \frac{\frac{(x+3)^{n+1}}{2^{n+1}(n+1)!}}{\frac{(x+3)^n}{2^n n!}} \right| = \left| \frac{x+3}{2(n+1)} \right| \to 0.$$

2. (6) What is the value of the improper integral $\int_e^{\infty} \frac{dx}{x(\ln x)^2}$?

A. -1

B. 0

C. $1/e$

D. 1

E. the integral diverges

$$= \lim_{b\to\infty} \int_e^b \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \int_{x=e}^{x=b} \frac{du}{u^2},$$

where $u = \ln x$, $du = \frac{1}{x}\,dx$,

$$= \lim_{b\to\infty} -\frac{1}{u}\Big|_{x=e}^{x=b} = \lim_{b\to\infty} \frac{-1}{\ln b} + 1.$$

3. (6) What do the level curves of the function $f(x, y) = 3x^2 + 4y^2$ look like?

A. circles

B. ellipses

C. diamonds

D. parabolas

E. waves

$3x^2 + 4y^2 = c$ is the equation of an ellipse.

4. (6) What is the distance from the point $(2, 3, 4)$ to the x-axis?

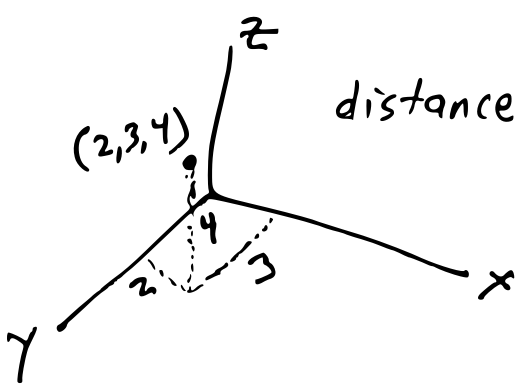
A. 2

B. 3

C. $\sqrt{13}$

D. 4

E. 5



distance $= \sqrt{3^2 + 4^2} = 5$

5. (6) Suppose $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$, and $\mathbf{a} \times \mathbf{b} = 0$. What is $\mathbf{a} \cdot \mathbf{b}$?

$\vec{a} \times \vec{b} = \vec{0}$, so they are parallel, but they could point in the same or opposite directions.

A. 0

B. 2

C. 3

D. 6

E. not enough information to decide

So, $\vec{a} \cdot \vec{b} = 6$ or -6.

6. (6) Suppose that $\dfrac{n+7}{2-4n} \leq a_n \leq \dfrac{n^2-n}{(2n+3)^2}$ for all n. What is $\lim\limits_{n\to\infty} a_n$?

$\downarrow \quad -\frac{1}{4} \qquad \downarrow \quad \frac{1}{4}$

A. $-1/4$

B. 0

C. $1/4$

D. $1/2$

E. not enough information to decide

7. (6) If (a, b) is a critical point of the function $f(x, y)$ and $f_{xx}(a, b) > 0$, $f_{xy}(a, b) = 0$, and $f_{yy}(a, b) > 0$, what kind of critical point is (a, b)? (You may refer to the Second Derivative Test at the beginning of the other test booklet.)

A. local minimum

B. local maximum

C. saddle point

D. not enough information to decide

8. (6) If (a, b) is a critical point of the function $f(x, y)$ and $f_{xx}(a, b) > 0$, $f_{xy}(a, b) > 0$, and $f_{yy}(a, b) > 0$, what kind of critical point is (a, b)? (You may refer to the Second Derivative Test at the beginning of the other test booklet.)

A. local minimum

B. local maximum

C. saddle point

D. not enough information to decide

9. (6) Consider the series $\sum_{n=10}^{\infty} \frac{1}{n^{\ln \ln n}}$. Which of the following arguments is correct?

A. this series diverges by the Test for Divergence

B. this series diverges by comparison to $\sum \frac{1}{n}$

C. this series converges by comparison to $\sum \frac{1}{n}$

D. this series diverges by comparison to $\sum \frac{1}{n^2}$

E. this series converges by comparison to $\sum \frac{1}{n^2}$

For large n, $\ln \ln n \geq 2$.

10. (6) Suppose that $\nabla f = \langle 2xy + 3x^2y^2, x^2 + 2x^3y \rangle$ and $f(1, 2) = 3$. What is the tangent plane to the surface $z = f(x, y)$ at the point $(1, 2)$?

A. $z - 3 = (x - 1)/16 + (y - 2)/5$

B. $z - 3 = 16(x - 1) + 5(y - 2)$

C. $z - 3 = \langle 16, 5 \rangle t + \langle 1, 2 \rangle$

D. $z - 3 = (x - 16) + (y - 5)/2$

E. $z - 3 = (x - 16) + 2(y - 5)$

11. (6) Suppose $\text{proj}_{\mathbf{b}}\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$ and that the angle between $\mathbf{a}$ and $\mathbf{b}$ is obtuse. What is $\text{comp}_{\mathbf{b}}\mathbf{a}$?

Since the angle is obtuse,

$$\text{comp}_{\vec{b}}\vec{a} = -|\text{proj}_{\vec{b}}\vec{a}|$$

$$= -\sqrt{3^2 + 4^2}$$

$$= -5$$

A. 4

B. 5

C. -5

D. $4\mathbf{i} - 3\mathbf{j}$

E. $-3\mathbf{i} + 4\mathbf{j}$

12. (6) Which of the following vectors is parallel to the plane $2x - 3y + 4z = 10$?

We want $\vec{v} \cdot \langle 2, -3, 4\rangle = 0$.

$\langle 3, 2, 0\rangle \cdot \langle 2, -3, 4\rangle = 6 - 6 = 0$.

A. $\langle 1, -1, 1\rangle$

B. $\langle 4, -6, 8\rangle$

C. $\langle 6, 2, 1\rangle$

D. $\langle 3, 2, 0\rangle$

E. $\langle -1, -1, 2\rangle$

13. (6) Suppose that $f(x) = \sum_{n=0}^{\infty} \frac{n^2(x-3)^n}{2^n}$ for $|x-3| < 2$. What is $f^{(38)}(3)$? (The 38th derivative of f at 3.)

A. $\frac{38^2}{2^{38}\,38!}$

B. $\frac{38^2}{2^{38}}$

C. $\frac{38^2\,38!}{2^{38}}$

D. $\frac{38}{2^{38}}$

E. $\frac{38!}{2^{38}}$

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n$, so

$\frac{f^{(38)}(3)}{38!}(x-3)^n = \frac{38^2}{2^{38}}(x-3)^{38}$, so

$f^{(38)}(3) = \frac{38!\,38^2}{2^{38}}$

14. (6) What is the domain of the function $\frac{3x^2+2y^3}{\sqrt{x^2+y^2-4}}$?

A. the inside of a circle (including the circle)

B. the inside of a circle (not including the circle)

C. the outside of a circle (including the circle)

D. the outside of a circle (not including the circle)

E. the entire plane, except a circle

Need $x^2+y^2-4>0$,
so $x^2+y^2>4$.

15. (6) Suppose that $\nabla f = \langle 2xe^{2y}, 2x^2e^{2y} + 3y^2\rangle$. In which of these directions is $D_{\mathbf{u}}f$ maximized at the point $(2,0)$?

A. the x direction

B. the y direction

C. $\langle -4, -8\rangle$

D. $\langle 1/\sqrt{5}, 2/\sqrt{5}\rangle$

E. $\langle -1/\sqrt{5}, -2/\sqrt{5}\rangle$

$\nabla f(2,0) = \langle 4, 8\rangle$.

Direction $= \dfrac{\langle 4, 8\rangle}{\sqrt{96}} = \left\langle \dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}} \right\rangle$