3.2. MANIPULATION AND DERIVATION

We concluded the previous section by noting that power series can be differentiated and integrated term-by-term (within their radii of convergence). This is quite a strong property of power series (which does not hold for series in general). We begin this section by showing that term-by-term differentiation and integration can be used to *find* power series.

Our starting point will always (in this section, at least) be a geometric series of some kind, the simplest example of such being

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } |x| < 1,$$

which we know from our study of geometric series in Section 2.3. By differentiating (term-by-term) both sides of this equation, we obtain our first new power series:

$$\sum_{n=0}^{\infty} nx^{n-1} = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2} \text{ for } |x| < 1.$$

This is one of the three basic forms of power series we derive in this section. The other two are given in Examples 1 and 2.

Example 1. Find the power series centered at $x = 0$ for $\ln(1+x)$ and its radius of convergence.

Solution. Recall that $\ln(1+x)$ is the antiderivative of $1/(1+x)$:

$$\int \frac{1}{1+x}\,dx = \ln(1+x).$$

Furthermore, we can express $1/(1+x)$ as a geometric power series (for $|x| < 1$):

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n.$$

Therefore, all we have to do to get the power series for $\ln(1+x)$ is integrate this series term-by-term,

$$\begin{aligned} \ln(1+x) &= \int \frac{1}{1+x}\,dx \\ &= \int \sum_{n=0}^{\infty}(-1)^n x^n \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} + C. \end{aligned}$$

But what is C, the constant of integration? To find C we substitute $x = 0$ into both sides. We know that $\ln(1+0) = \ln 1 = 0$, so $C = 0$. This gives

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}.$$

The geometric series we integrated had radius of convergence $R = 1$, so the radius of convergence of this series for $\ln(1+x)$ is also $R = 1$. ●

The power series for $\ln(1+x)$ that we found in Example 1 is known as the *Mercator series*, after Nicholas Mercator (1620–1687). Note that by substituting $x = 1$ into this series, we obtain

$$\ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \ ?$$

This seems to indicate that the sum of the alternating harmonic series is $\ln 2$. However, there is a problem with this line of reasoning: term-by-term integration is only guaranteed to work *inside* the interval of convergence, and $x = 1$ is an endpoint of the interval of convergence for the Mercator series. Nevertheless, this computation can be made rigorous, as shown by Abel, see Exercise 33. (Another proof of this result, using Euler's constant γ is given in Exercises 46 and 47 of Section 2.4.)

We move on to another example of using integration to derive a power series.

Example 2. Find the power series centered at $x = 0$ for $\arctan x$ and its radius of convergence.

Solution. For this we need to recall that

$$\arctan x = \int \frac{1}{1+x^2}\, dx.$$

Again, we can write $1/(1+x^2)$ as a geometric power series (for $|x| < 1$):

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Now we integrate this series term-by-term:

$$\begin{aligned} \arctan x &= \int \frac{1}{1+x^2}\, dx \\ &= \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C. \end{aligned}$$

Finally, we substitute $x = 0$ into both sides of this equation to see that $C = 0$, giving

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \text{ for } |x| < 1.$$

The geometric series we integrated had radius of convergence $R = 1$, so the radius of convergence of this series for $\arctan x$ is also $R = 1$. ●

As in Example 1, this series suggests an intriguing equality:

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \text{ ?}$$

But once again, $x = 1$ lies on the endpoint of the interval of convergence for this series, so this equation does not necessarily follow from what we have done. Nevertheless, as with the previous example, this can be made precise using Abel's Theorem, see Exercise 34. (This is called the GregoryLeibniz formula for π, after Gottfried Leibniz (1646–1716) and James Gregory (1638–1675)).

But what if we wanted power series for $\ln(1 + 3x^2)$ or $\arctan 2x^3$? We could write them as integrals and then integrate some form of geometric power series as in Examples 1 and 2, but this is tedious and error-prone. More worryingly, what about more complicated functions like $\ln(1 + x)/(1 + 2x)$?

Just as with differentiation and integration, it turns out that *within their radii of convergence* we may treat power series just like polynomials when

- substituting,
- multiplying, and
- dividing.

Here even stating the theorems is technical; we instead illustrate the point with examples.

Example 3. Find the power series centered at $x = 0$ for $\dfrac{1}{(1 - 8x^3)^2}$.

Solution. We know from the beginning of the section that

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} \text{ for } |x| < 1,$$

so to get the power series for $1/(1 - 8x^3)^2$, we simply replace x with $8x^3$:

$$\frac{1}{(1-8x^3)^2} = \sum_{n=0}^{\infty} n \left(8x^3\right)^{n-1} = \sum_{n=0}^{\infty} n8^{n-1}x^{3n-3}.$$

As the power series for $1/(1-x)^2$ held when $|x| < 1$, this power series holds when $|8x^3| < 1$, which simplifies to $|x| < 1/2$. ●

Example 4. Find the power series centered at $x = 0$ for $\ln(4 + 3x^2)$.

Solution. First we need to get the function in the form $\ln(1 + \text{something})$:

$$\ln(4 + 3x^2) = \ln\left(4 \cdot \left(1 + \frac{3x^2}{4}\right)\right) = \ln 4 + \ln\left(1 + \frac{3x^2}{4}\right).$$

Now we substitute $3x^2/4$ into the power series we found for $\ln(1 + x)$ in Example 1:

$$\ln 4 + \ln\left(1 + \frac{3x^2}{4}\right) = \ln 4 + \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{3x^2}{4}\right)^{n+1}}{n+1} = \ln 4 + \sum_{n=0}^{\infty} (-1)^n \frac{3^{n+1} x^{2n+2}}{4^{n+1}(n+1)}.$$

Note that our series for $\ln(1 + x)$ was valid for $|x| < 1$, so this new series is valid for $|3x^2/4| < 1$, or $|x| < \sqrt{4/3}$. ●

Example 5. Find the power series centered at $x = 0$ for $\displaystyle\int \frac{\arctan 2x^3}{x^3}\,dx$.

Solution. First we substitute $2x^3$ into our power series for $\arctan x$ from Example 2 to find a power series for $\arctan 2x^3$:

$$\arctan 2x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{(2x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{6n+3}}{2n+1}.$$

Now we divide each of the terms of this series by x^3 and integrate term-by-term:

$$\begin{aligned}
\int \frac{\arctan 2x^3}{x^3}\,dx &= \int \frac{\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{6n+3}}{2n+1}}{x^3}\,dx \\
&= \int \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{6n}}{2n+1}\,dx \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{6n+1}}{(2n+1)(6n+1)} + C.
\end{aligned}$$

Since our series for $\arctan x$ was valid for $|x| < 1$, this series is valid for $|2x^3| < 1$, or $|x| < \sqrt[3]{1/2}$. ●

In practice, it can be quite tedious to find many coefficients by multiplication. However, the first few coefficients are the most important.

Example 6. Compute the first four nonzero terms of the power series for $\dfrac{\ln(1+x)}{1+2x}$.

Solution. We simply need to multiply the geometric power series for $1/(1+2x)$ with the power series for $\ln(1+x)$ that we found in Example 1:

$$\begin{aligned}\frac{\ln(1+x)}{1-2x} &= \left(1-2x+4x^2-8x^3+\cdots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)\\ &= x-\frac{5x^2}{2}+\frac{16x^3}{3}-\frac{131x^4}{12}+\cdots\end{aligned}$$

Since the series for $1/(1+2x)$ has radius of convergence $1/2$ and the series for $\ln(1+x)$ has radius of convergence 1, the radius of convergence of their product is the minimum of these two values, $1/2$. ●

In the next section we use division to derive the power series for $\tan x$. In order to show how division works in this section, we repeat the previous example, dividing instead of multiplying.

Example 7. Use division to compute the first four nonzero terms of the power series for the function $f(x) = \dfrac{\ln(1+x)}{1+2x}$.

Solution. We use long division to divide the power series for $\ln(1+x)$ by $1+2x$:

$$\begin{array}{r|llllll}
 & x - & 5x^2/2 & + 16x^3/3 - & 131x^4/12 & + \cdots \\ \hline
1+2x & x - & x^2/2 & + x^3/3 - & x^4/4 & + \cdots \\
 & x + & 2x^2 & & & \\ \hline
 & & -5x^2/2 + & x^3/3 - & x^4/4 & + \cdots \\
 & & -5x^2/2 - & 5x^3 & & \\ \hline
 & & & 16x^3/3 - & x^4/4 & + \cdots \\
 & & & 16x^3/3 + & 32x^4/3 & + \cdots \\ \hline
 & & & & -131x^4/12 & + \cdots
\end{array}$$

Note that this agrees with our computation in the previous example. ●

EXERCISES FOR SECTION 3.2

Find power series centered at $x = 0$ for the functions in Exercises 1–16 and give their radii of convergence.

1. $f(x) = \dfrac{1}{1+x}$

2. $f(x) = \dfrac{2}{3+x}$

3. $f(x) = \dfrac{3}{1-x^3}$

4. $f(x) = \dfrac{4}{2x^3+3}$

5. $f(x) = \dfrac{2}{(1+x)^2}$

6. $f(x) = \dfrac{x}{(1+x)^2}$

7. $f(x) = \dfrac{x^2}{(1+x^5)^2}$

8. $f(x) = \dfrac{2}{(4-2x^2)^2}$

9. $f(x) = \ln(1-x)$

10. $f(x) = \ln(1-2x^3)$

11. $f(x) = \ln(e - e^2x^2)$

12. $f(x) = x^2 \arctan(3x^3)$

13. $f(x) = \arctan(x/2)$

14. $f(x) = \dfrac{\arctan(x)}{1+x}$

15. $f(x) = (1+x)\ln(1+x)$

16. $f(x) = \ln(1+x)^{1+x} + \ln(1-x)^{1-x}$

17. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n} = \ln 2$. *Hint:* try substituting an appropriate value of x into the series for $\ln(1+x)$.

18. Use division of power series to give another proof that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$.

Exercises 19 and 20 explore the Fibonacci numbers.

19. Define

$$f(x) = \sum_{n=0}^{\infty} f_n x^n,$$

where f_n denotes the nth Fibonacci number. Use the recurrence relation and initial conditions for $\{f_n\}$ to show that

$$f(x) = 1 + xf(x) + x^2 f(x),$$

and derive from this that

$$f(x) = \frac{1}{1-x-x^2}.$$

20. Use Exercise 19, partial fractions, and geometric series to derive Binet's formula.

Use partial fractions and geometric series to find formulas for the coefficients of the functions in Exercises 21–24.

21. $f(x) = \dfrac{x}{1-5x+6x^2}$

22. $f(x) = \dfrac{2-5x}{1-5x+6x^2}$

23. $f(x) = \dfrac{x}{1-3x-2x^2}$

24. $f(x) = \dfrac{244-246x}{1-4x+3x^2}$

25. Using the fact that

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ for } |x| < 1,$$

show that

$$\lim_{x \to 1^-} \sum_{n=1}^{\infty} \frac{x^n}{n} = \infty,$$

thereby proving (again) that the harmonic series diverges.

♦ **26.** Let $\sum a_n$ be a series with partial sums $\{s_n\}$ and suppose that the function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for $|x| < 1$. Prove that

$$\sum_{n=0}^{\infty} s_n x^n = \frac{f(x)}{1-x}.$$

27. The *harmonic numbers* $\{H_n\}$ are defined by $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$. Show that

$$\sum_{n=1}^{\infty} H_n x^n = \frac{1}{1-x} \ln \frac{1}{1-x}.$$

The sequence $\{a_n\}$ is said to be *Abel summable* to L if $\sum a_n x^n$ converges on (at least) the interval $[0, 1)$ and

$$\lim_{x \to 1^-} a_n x^n = L.$$

(Note that the first term of these sequences is a_0.) In Exercises 28–32 consider Abel summability. This concept is due to Niels Henrik Abel (1802–1829).

28. Show that the sequence $\{2^{-n}\}$ is Abel summable to 2.

29. Show that $\{(-1)^n\}$ is Abel summable to $1/2$. (C.f. Exercise 50 from Section 2.2.)

30. Show that $\left\{\dfrac{(-1)^n}{n+1}\right\}$ is Abel summable to $\ln 2$.

31. Show that $\{(-1)^{n+1} n\}$ is Abel summable to $1/4$. (C.f. Exercise 51 in Section 2.2.)

32. Show that $\{(-1)^n n(n-1)\}$ is Abel summable to $1/4$.

Abel's Theorem guarantees that convergent series are Abel summable to their true values:

Abel's Theorem. If the series $\sum a_n$ converges to a finite value L, then $\{a_n\}$ is Abel summable to L.

Assume the truth of Abel's Theorem in Exercises 33 and 34.

33. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$

34. Show that $\displaystyle 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \pi.$

35. Exercise 33 shows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$$

(this is also proved in Exercises 46 and 47 of Section 2.4), while Exercise 17 gives a different series that converges to $\ln 2$. Which of these two series converges "faster"?

In 1995, after a several month long search by computer, Bailey, Borwein, and Plouffe discovered a remarkable formula for π which allows one to compute *any* binary or hexadecimal digit of π without computing any of the digits that come before it. For details about how the formula was discovered, we refer the reader to the book *Mathematics by Experiment* by Borwein and Bailey. Exercises 36–38 establish the formula.

36. Show that

$$\int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1 - x^8}\, dx = \pi.$$

Hint: Once we substitute $u = \sqrt{2}x$, the integral becomes

$$\int_0^1 \frac{16u - 16}{u^4 - 2u^3 + 4u - 4}\, du,$$

which, using partial fractions, is equal to

$$\int_0^1 \frac{4u}{u^2 - 2}\, du - \int_0^1 \frac{4u - 8}{u^2 - 2u + 2}\, du.$$

The first integral requires another substitution, while the second must be split into two integrals. One can be evaluated by substitution, the other can be done by writing $u^2 - 2u + 2$ as $(u-1)^2 + 1$.

37. Let k be a fixed integer less than 8. Show that

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1 - x^8}\, dx = \frac{1}{2^{k/2}} \sum_{n=0}^{\infty} \frac{1}{16^n (8n + k)}.$$

38. Use Exercises 36 and 37 to show that π is equal to

$$\sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right).$$

This is the Bailey-Borwein-Plouffe formula for π.

ANSWERS TO SELECTED EXERCISES, SECTION 3.2

1. $\sum_{n=0}^{\infty}(-1)^n x^n$

3. $\sum_{n=0}^{\infty}3x^{3n}$

5. $\sum_{n=0}^{\infty}2n(-1)^n x^{n-1}$

7. $\sum_{n=0}^{\infty}(-1)^n n x^{5n-3}$

9. $\sum_{n=0}^{\infty}-\frac{x^{n+1}}{n+1}$

11. $1-\sum_{n=0}^{\infty}\frac{e^{n+1}x^{2n+2}}{n+1}$

13. $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^{2n+1}(2n+1)}$