3. POWER SERIES

3.1. SERIES AS FUNCTIONS

At the end of Chapter 1 we saw that Taylor polynomials of "infinite degree" might be valuable for approximating functions. First we needed to understand what it meant to add infinitely many numbers together, a notion that we formalized and studied in Chapter 2. With these prerequisites covered, we return to the analysis of functions, beginning by studying *power series*, which are series involving powers of x, such as

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots ^{\dagger}.$$

This power series is *centered* at $x = 0$. Our definition below is slightly more general.

> **Power Series.** A *power series centered at* $x = a$ is a series of the form
>
> $$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots .$$

The first question we should ask is:

Given a power series, for what values of x does it converge?

As the next three examples show, the techniques we have developed to analyze series are capable of answering this question as well.

Example 1. Find the values of x for which the power series $\sum_{n=0}^{\infty} n!(x+2)^n$ converges and plot them on a number line.

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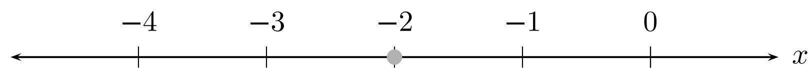
†The reader may have noticed that we have switched (mostly) from sums starting at $n = 1$ to sums starting at $n = 0$. This is because when dealing with regular series, it is natural to index the first term as a_1, while with power series it is more convenient to index the terms based on the power of x.

Solution. We test the series for absolute convergence using the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!(x+2)^{n+1}}{n!(x+2)^n}\right| = |(n+1)(x+2)| \to \infty \text{ unless } x = -2.$$

Therefore the series converges only when $x = -2$, so our plot is a single point,



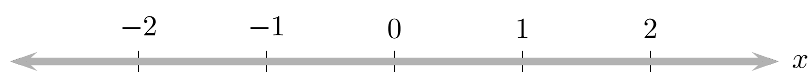
showing that the power series converges only at $x = -2$. ●

Example 2. For what values of x does the power series $\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$ converge?

Solution. We again test the series for absolute convergence using the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-1)^{n+1}}{(n+1)!}}{\frac{(x-1)^n}{n!}}\right| = \left|\frac{x-1}{n+1}\right| \to 0 \text{ for all } x.$$

Therefore this series converges (absolutely) for every x, so our number line contains all real numbers,



We can also write this as the interval $(-\infty, \infty)$, or we may simply express it as the set of all real numbers, $\mathbb{R}$. ●

Our third and final example is a bit more interesting.

Example 3. For what values of x does the power series $\sum_{n=0}^{\infty} \frac{(x+3)^n}{(n+1)4^n}$ converge?

Solution. Again we begin by testing the series for absolute convergence with the Ratio Test, although in this case we will need to work more afterward:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x+3)^{n+1}}{(n+2)4^{n+1}}}{\frac{(x+3)^n}{(n+1)4^n}}\right| = \left|\frac{x+3}{4} \cdot \frac{n+2}{n+1}\right| \to \left|\frac{x+3}{4}\right|.$$

For what values of x is $\left|\frac{x+3}{4}\right| < 1$? This inequality can be rewritten as

$$-1 < \frac{x+3}{4} < 1,$$

or, simplifying,

$$-4 < x + 3 < 4,$$

so the given power series converges by the Ratio Test if $-7 < x < 1$, or in other words, if x lies in the interval $(-7, -1)$. The power series diverges if $x < -7$ or $x > 1$. But when $x = -7$ or $x = 1$, the Ratio Test is inconclusive, so we have to test these *endpoints* individually. This is typical for power series, not specific to this example.

Plugging in $x = -7$, our series simplifies to

$$\sum_{n=0}^{\infty} \frac{(-4)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

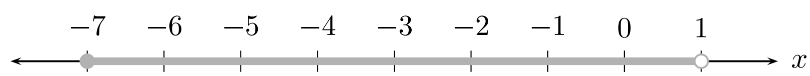
Since this is the alternating harmonic series, we know that it converges (conditionally). So, our power series converges at $x = -7$.

Plugging in $x = 1$, our series simplifies to

$$\sum_{n=0}^{\infty} \frac{4^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{1}{n+1}.$$

This is the harmonic series which we know diverges, so our power series diverges at $x = 1$.

Putting this all together, the given power series converges if and only if $-7 \leqslant x < 1$, which we can also write as the interval $[-7, 1)$. The number line plot of this interval is:



(Here the closed circle means that $x = -7$ is included, while the open circle means that $x = -1$ is excluded.) ●

These examples have demonstrated three different types of convergence. As our next theorem shows, *every* power series exhibits one of these three behaviors.

Radius Theorem. Every power series $\sum c_n(x-a)^n$ satisfies one of the following:

(1) The series converges only when $x = a$, and this convergence is absolute.

(2) The series converges for all x, and this convergence is absolute.

(3) There is a number $R > 0$ such that the series converges absolutely when $|x - a| < R$ and diverges when $|x - a| > R$. Note that the series may converge absolutely, converge conditionally, or diverge when $|x - a| = R$.

The proof of the Radius Theorem is outlined in Exercises 40–45.

Case (1) of this theorem holds when the coefficients c_n are "large", while case (2) holds when these coefficients are "small". Case (3) holds for coefficients which lie somewhere in between these two extremes. Note that for series which satisfy case (3), the interval of convergence is *centered at* $x = a$. We call the number R in this case the *radius of convergence*. (In case (1) we might say that the radius of convergence is 0, while in case (2) we might say that it is ∞.)

When case (3) holds, there are four possibilities for the interval of convergence:

$$(a-R, a+R), \quad (a-R, a+R], \quad [a-R, a+R), \quad [a-R, a+R].$$

In order to decide which of these is the interval of convergence, we must test the endpoints one-by-one, as we did in Example 3. Therefore the general procedure for determining the interval of convergence of a given power series is:

1. Identity the center of the power series.
2. Use the Ratio Test to determine the radius of convergence. (There are rare instances in which the Ratio Test is not sufficient, in which case the Root Test should be used instead, see Exercises 20–26 of Section 2.6.)
3. If the series has a positive, finite radius of convergence (case (3)), then we need to test the endpoints $a - R$ and $a + R$. These two series may be tested with any method from the last chapter.

When a power series converges, it defines a function of x, so our next question is:

What can we say about functions defined as power series?

The short answer is that *inside its radius of convergence*, a power series can be treated like a long polynomial. In particular, we can differentiate power series like polynomials:

Term-by-Term Differentiation. Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

converges for all x in the interval $(a-R, a+R)$. Then f is differentiable for all values of x in the interval $(a-R, a+R)$, and

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots .$$

In particular, the radius of convergence of $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ is at least R.

This theorem says quite a lot about the behavior of a power series inside its radius of convergence. Not only can we differentiate such a power series, but the derivative has *at least as large a radius of convergence*. Well then, there's nothing stopping us from taking the derivative of this derivative, and so on. Therefore, inside its radius of convergence, a power series defines an *infinitely differentiable*, or *smooth*, function of x. Such functions are extremely well-behaved. For one, remember that in order to be differentiable, a function must first be continuous, so inside its radius of convergence, a power series defines a continuous function.

Integration can be handled the same way, by treating a power series as a long polynomial inside its radius of convergence.

Term-by-Term Integration. Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

converges for all x in the interval $(a-R, a+R)$. Then

$$\begin{aligned} \int f(x)\,dx &= \sum_{n=1}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C \\ &= c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots + C, \end{aligned}$$

and this series converges for all x in the interval $(a-R, a+R)$.

Term-by-term differentiation and integration should *not* seem obvious, and their justification takes quite a bit of work, even in more advanced courses. For now, we take them for granted.

EXERCISES FOR SECTION 3.1

Find the intervals of convergence of the series in Exercises 1–10.

1. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt[7]{n}}$

2. $\sum_{n=0}^{\infty} \frac{3^n x^n}{(n+2)^3}$

3. $\sum_{n=0}^{\infty} \frac{3^n (x+2)^n}{(n+2)^3}$

4. $\sum_{n=0}^{\infty} \frac{(4x)^n}{n^4}$

5. $\sum_{n=0}^{\infty} \frac{(-4x)^n}{n^4}$

6. $\sum_{n=0}^{\infty} \frac{(-4x+2)^n}{n^4}$

7. $\sum_{n=0}^{\infty} x^n n!$

8. $\sum_{n=0}^{\infty} \frac{(x-2)^n n!}{2^n}$

9. $\sum_{n=0}^{\infty} \left(\frac{x+2}{3n}\right)^n$

10. $\sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{n!} x^n$

In Exercises 11–20, construct a power series with the given intervals of converge, or explain why one does not exist.

11. $(-2, 2)$

12. $(-4, 0)$

13. $[0, 2]$

14. $(-\infty, \infty)$

15. $[0, \infty)$

16. $(1, \infty)$

17. $[3, 7)$

18. $(3, 7]$

19. $(-\infty, 2]$

20. $\{7\}$

Rewrite the expressions in Exercises 21–24 as series in which the generic term involves x^n.

21. $\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$

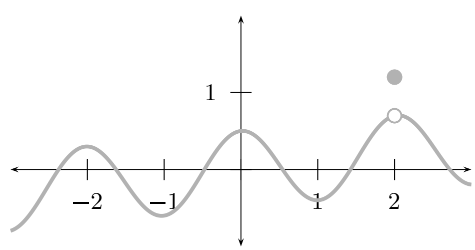
22. $\sum_{n=0}^{\infty} c_n x^{n+3}$

23. $\sum_{n=1}^{\infty} nc_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n$

24. $x \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$

Explain why none of the functions plotted in Exercises 25–28 are equal to power series on the interval $(-3, 3)$.

25.



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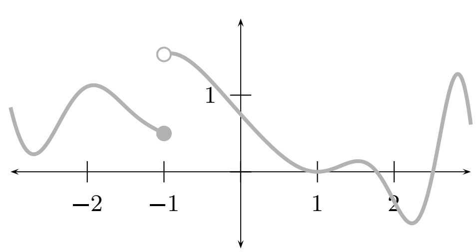

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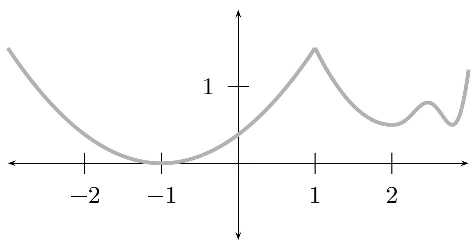

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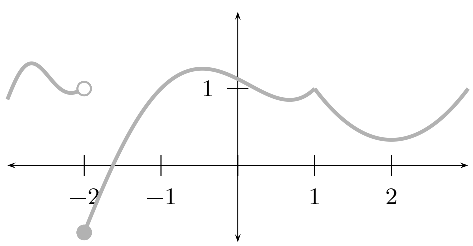


29. Explain why the radius of convergence of the power series for $f(x) = \tan x$ centered at $a = 0$ is at most $\pi/2$.

30. Explain why a power series can converge conditionally for at most two points.

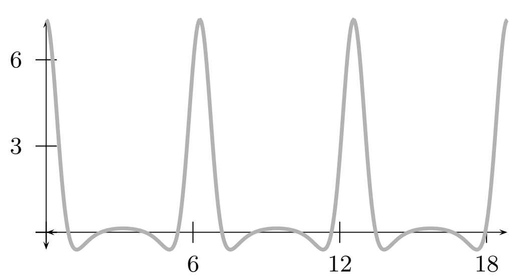
Exercises 31–33 concern the series

$$\sum_{n=0}^{\infty} \frac{2^n \cos nx}{n!}.$$

Note that this is *not* a power series. Below, the 100th partial sum,

$$\sum_{n=0}^{100} \frac{2^n \cos nx}{n!}$$

of this series is plotted.



31. Show that this series converges for all x.

32. Does this series define a periodic function of x, as the plot above seems to demonstrate?

33. Verify that the actual series is within $1/100$ of the partial sum plotted above for all values of x.

34. Suppose you know, from using the Ratio Test, that the radius of convergence of $\sum c_n x^n$ is $R = 6$. What is the radius of convergence of $\sum c_n n^3 x^n$?

35. Suppose you know, from using the Ratio Test, that the radius of convergence of $\sum c_n x^n$ is $R = 6$. What is the radius of convergence of $\sum c_n x^n / 3^n$?

36. Suppose that the radius of convergence of $\sum c_n x^n$ is $R \geqslant 1$. Then what is the radius of convergence of $\sum s_n x^n$ where $s_n = c_0 + c_1 + \cdots + c_n$?

37. Suppose that the radius of convergence of $\sum c_n x^n$ is $R < 1$. Then what is the radius of convergence of $\sum s_n x^n$ where $s_n = c_0 + c_1 + \cdots + c_n$?

Use the bounds given by Exercises 50 and 50 of Section 2.4 to find the radii of convergence of the series in Exercises 38 and 39.

38. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n} x^n$

39. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n!)}{\sqrt{3n^2 + 2n + 1}} x^n$

Exercises 40–45 detail the proof of the Radius Theorem. For simplicity, we assume that the series is centered at 0, that is, that $a = 0$, but the proof easily extends to other centers by making a change of variables, setting $y = x - a$.

40. Suppose that the power series $\sum c_n x^n$ converges at $x = s$. Prove that there is an integer N so that $|c_n| < 1/|s|^n$ for all $n \geqslant N$.

41. With N as in the previous exercise, prove that if $|x| < |s|$ and $n \geqslant N$, $|c_n x^n| < |x/s|^n$.

42. Using the previous two exercises and the Comparison Test, prove that if the power series $\sum c_n x^n$ converges at $x = s$ then it converges absolutely whenever $|x| < |s|$.

43. Prove that if the power series $\sum c_n x^n$ diverges at $x = t$ then it diverges whenever $|x| > |t|$.

44. Define C to be the set of values of x for which $\sum c_n x^n$ converges. Prove that either C contains all real numbers or C is bounded.

45. Use the fact that every bounded set of real numbers has a least upper bound (this is called the Completeness Property) to prove the Radius Theorem. (The least upper bound b of the set C is the least number such that $c \leqslant b$ for all $c \in C$.)

ANSWERS TO SELECTED EXERCISES, SECTION 3.1

1. $[-1, 1)$
3. $[-7/3, -5/3]$
5. $[-1/4, 1/4]$
7. Converges only for $x = 0$
9. Converges for all real numbers
11. One example is $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$
13. One example is $\sum_{n=0}^{\infty} \frac{(x-1)^n}{n^2}$
15. No such power series exists, by the Radius Theorem
17. One example is $\sum_{n=0}^{\infty} \frac{(x-5)^n}{\sqrt{n}2^n}$
19. No such power series exists, by the Radius Theorem
21. $\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$
23. $\sum_{n=0}^{\infty} (n+1)c_{n+1}x^n + \sum_{n=1}^{\infty} 2a_{n-1}x^n$
25. Because the function has a discontinuity at $x = 2$
27. Because the function has a sharp corner at $x = 1$ (so its first derivative is not defined there)
29. Because $\tan x$ has a vertical asymptote at $x = \pi/2$