2.5. THE COMPARISON TEST

We began our systematic study of series with geometric series, proving the

- Geometric Series Test: $\sum ar^n$ converges if and only if $|r| < 1$.

Then in the last section we compared series to integrals in order to determine if they converge or diverge, and established the

- p-Series Test: $\sum 1/n^p$ converges if and only if $p > 1$.

In this section we study another type of comparison where we compare series to other series to determine convergence. The general principle is this:

- if a positive series is bigger than a positive divergent series, then it diverges, and
- if a positive series is smaller than a positive convergent series, then it converges.

For example, in the last section (Example 1) we showed that $\sum 1/n^2$ converges using the Integral Test. Then we used the Integral Test again (Example 2) to show that $\sum 1/n^2+1$ converges. But, $1/n^2+1$ is smaller than $1/n^2$ for all $n \geqslant 1$, so the convergence of $\sum 1/n^2+1$ is guaranteed by the convergence of $\sum 1/n^2$. While this approach should seem intuitively clear and simple, we caution the reader that it takes a lot of practice to become comfortable with comparisons. We state the formal test below.

The Comparison Test. Suppose that $0 \leqslant a_n \leqslant b_n$ for sufficiently large n.

- If $\sum a_n$ diverges, then $\sum b_n$ also diverges.
- If $\sum b_n$ converges, then $\sum a_n$ also converges.

Before presenting the proof of the Comparison Test, note the phrase "sufficiently large n". By "$0 \leqslant a_n \leqslant b_n$ for sufficiently large n", we mean that there is some number N such that $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant N$. This is just a formal way to say that we only care about tails, and should remind the reader of the Tail Observation from Section 2.2.

Proof. Suppose that for *all* n, $0 \leqslant a_n \leqslant b_n$. This seems slightly weaker than the result we have claimed, but the full result will then follow either by the Tail Observation of Section 2.2 or by an easy adaptation of this proof.

Let s_n denote the nth partial number of $\{a_n\}$ and t_n denote the nth partial sum of $\{b_n\}$, so

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n, \\ t_n &= b_1 + b_2 + \cdots + b_n. \end{aligned}$$

From our hypotheses (that $0 \leqslant a_n \leqslant b_n$ for all n), we know that $s_n \leqslant t_n$ for all n.

First suppose that $\sum a_n$ diverges. Because the terms a_n are nonnegative, the only way that $\sum a_n$ can diverge is if $s_n \to \infty$ as $n \to \infty$ (why?). Therefore the larger partial sums t_n must also tend to ∞ as $n \to \infty$, so the series $\sum b_n$ diverges as well.

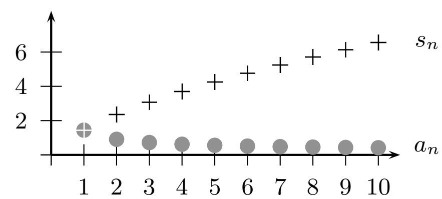
Now suppose that $\sum b_n$ converges, which implies by our definitions that $t_n \to \sum b_n$ as $n \to \infty$. The sequence $\{s_n\}$ is nonnegative and monotonically increasing because $a_n \geqslant 0$ for all n, and

$$0 \leqslant s_n \leqslant t_n \leqslant \sum b_n$$

so the sequence $\{s_n\}$ has a limit by the Monotone Convergence Theorem. This shows (again, by the definition of series summation) that the series $\sum a_n$ converges. □

The Comparison Test leaves open the question of what to compare series with. In practice, however, this choice is usually obvious, and we will almost always compare with a geometric series or a p-series. Our next four examples demonstrate the general technique.

Example 1. Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ converge or diverge?



Solution. First note that we probably shouldn't try to apply the Integral Test in this example — the function $1/\ln x$ has an antiderivative, but it has been proved that its antiderivative cannot be expressed in terms of elementary functions.

However, the Comparison Test is easy to apply in this case. Note that

$$\begin{array}{rcll} \ln n & \leqslant & n & \text{for } n \geqslant 2\text{, so} \\ {}^{1}\!/\!{}_{\ln n} & \geqslant & {}^{1}\!/\!{}_{n} & \text{for } n \geqslant 2. \end{array}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by comparison. ●

Example 2. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converge or diverge?

Solution. This example can be done with the Integral Test, but it's easier to use the Comparison Test. We know that $\ln n > 1$ for $n \geqslant 3$, so

$$\ln n/n \;\geqslant\; 1/n \quad \text{for } n \geqslant 3.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ must also diverge. ●

Example 3. Does the series $\sum_{n=1}^{\infty} \left(\frac{\cos(n)}{n}\right)^2$ converge or diverge?

Solution. We can write this series as $\sum \cos^2(n)/n^2$. The numerator of this fraction, $\cos^2(n)$, is nonnegative for all n (this is important since we can't apply the Comparison Test to series with negative terms) and bounded by 1, so

$$\frac{\cos^2(n)}{n^2} \leqslant \frac{1}{n^2} \quad \text{for } n \geqslant 1.$$

Therefore since $\sum 1/n^2$ converges (it is a convergent p-series), the smaller series $\sum_{n=1}^{\infty} \left(\frac{\cos(n)}{n}\right)^2$ must converge as well. ●

Example 4. Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}}$ converge or diverge?

Solution. For $n \geqslant e^2 \approx 7.39$, $\ln n \geqslant 2$, so for these values of n,

$$1/n^{\ln n} \leqslant 1/n^2.$$

Since $\sum 1/n^2$ is a convergent p-series, $\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}}$ converges by comparison. ●

Sometimes the inequalities we need to apply the Comparison Test seem to go the wrong way. Consider for example the series

$$\sum_{n=1}^{\infty} \frac{1}{2n+1}$$

We would like to compare this series with the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{2n},$$

but the terms in our series seem to be *smaller* than the terms of $\sum 1/2n$. Therefore we cannot naively apply the Comparison Test in this case.

Example 5. Show that the series $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ diverges.

Solution. We have that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{2n+1} &= \frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\\ &\geqslant \frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots=\sum_{n=2}^{\infty}\frac{1}{2n}=\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n},\end{aligned}$$

so the series diverges by comparison to $\sum 1/n$. ●

Our next example displays a similar phenomenon. Note that $1/n^2-1 > 1/n^2$, but we are still able to compare the series.

Example 6. Show that the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2-1}$ converges.

Solution. Because $n^2-1 \geqslant (n-1)^2$, we have that

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{1}{n^2-1} &= \frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\cdots\\ &\leqslant \frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2},\end{aligned}$$

so the series converges by comparison to $\sum 1/n^2$. ●

Example 7. Show that the series $\displaystyle\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^4+7}}$ diverges.

Solution. We should expect this series to diverge, because the numerator is n and the denominator behaves like n^2, but the inequality goes the wrong way. By giving up a bit in the denominator, however, we get the desired conclusion:

$$\frac{n}{\sqrt{n^4+7}} \geqslant \frac{n}{\sqrt{n^4+7n^4}} = \frac{1}{\sqrt{8}n},$$

so the series we are interested in diverges by comparison to the harmonic series. ●

In Examples 5–7, we are really *reindexing* the series. This procedure is demonstrated more formally in the example below and in Exercises 25–28. Another method for dealing with such problems, known as the Limit Comparison Test, is discussed in Exercises 42–50.

Example 8. Show that the series $\displaystyle\sum_{n=2}^{\infty}\frac{n^2+3}{n^4-2}$ converges by reindexing the series with the substitution $m = n-1$.

Solution. We want to compare this series to the series given by its leading terms, $\sum n^2/n^4$ (or some multiple of this), but the comparison seems to go the wrong way. By setting $m = n - 1$, which is equivalent to $n = m + 1$, we have

$$\sum_{n=2}^{\infty} \frac{n^2+3}{n^4-2} = \sum_{m=1}^{\infty} \frac{(m+1)^2+3}{(m+1)^4-2} = \sum_{m=1}^{\infty} \frac{m^2+2m+4}{m^4+4m^2+6m^2+4m-1}.$$

(Note here the change in the lower bound, as in the previous example.) The inequality in the numerators (we want to compare m^2+2m+4 with m^2) still goes the wrong way, but we can take care of this by using a slightly different inequality:

$$m^2+2m+4 \leqslant m^2+2m^2+4m^2 = 7m^2$$

for $m \geqslant 1$. The inequality in the denominators does go the right way:

$$m^4+4m^2+6m^2+4m-1 \geqslant m^4.$$

Since we have made the numerators larger and the denominators smaller, we have made the fractions larger, and thus

$$\sum_{m=1}^{\infty} \frac{m^2+2m+4}{m^4+4m^2+6m^2+4m-1} \leqslant \sum_{m=1}^{\infty} \frac{7m^2}{m^4} = \sum_{m=1}^{\infty} \frac{7}{m^2},$$

which implies by the Comparison Test that the series in question converges, because $\sum 7/m^2 = 7\sum 1/m^2$ is a convergent p-series. ●

Our next example doesn't require reindexing, but does require a clever bound for $\ln n$. So far we have used the facts that $\ln n \leqslant n$ for $n \geqslant 2$ (in Example 1) and $\ln n \geqslant 1$ for $n \geqslant 3$ (in Example 2). In fact, a much stronger upper bound holds. Let p be any positive real number. Then by l'Hôpital's Rule, we have

$$\lim_{x\to\infty} \frac{\ln x}{x^p} = \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0.$$

Recalling a fact from Section 2.1, this means that

$$\lim_{n\to\infty} \frac{\ln n}{n^p} = 0$$

for *every* $p > 0$. This in turn means that for every $p > 0$, $\ln n \leqslant n^p$ for sufficiently large n, a handy fact to have around for comparisons, as we demonstrate next.

Example 9. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n \ln n}{\sqrt{(n+3)^5}}$ converge or diverge?

Solution. As we showed above, $\ln n \leqslant n^{1/4}$ for sufficiently large n (we could give a smaller bound, but $1/4$ is good enough here) and $(n+3)^5 \geqslant n^5$, we can use the comparison

$$\frac{n \ln n}{\sqrt{(n+3)^5}} \leqslant \frac{n^{1+1/4}}{n^{5/2}} = \frac{1}{n^{5/4}}.$$

Because $\sum 1/n^{5/4}$ is a convergent p-series, $\displaystyle\sum_{n=1}^{\infty} \frac{n \ln n}{\sqrt{(n+3)^5}}$ converges by the Comparison Test. ●

Our last example is considerably trickier than the previous examples. The reader should pay attention to the two themes it demonstrates: first, when dealing with a variable in an exponent, it is a good idea to use e and natural log, and second, no matter how slowly a function (such as $\ln \ln n$) goes to infinity, it must eventually grow larger than 2!

Example 10. Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ converge or diverge?

Solution. As the terms have a variable in the exponent, we first manipulate the using e and $\ln$:

$$(\ln n)^{\ln n} = e^{\ln\left(\ln n^{\ln n}\right)} = e^{\ln n \ln \ln n} = n^{\ln \ln n}.$$

We now need to test $\displaystyle\sum_{n=2} \frac{1}{n^{\ln \ln n}}$ for convergence. The approach from here on is similar to Example 4: for $n \geqslant e^{e^2} \approx 1618.18$ (i.e., for large n), we have $\ln \ln n \geqslant 2$, so

$$\frac{1}{n^{\ln \ln n}} \leqslant \frac{1}{n^2},$$

and thus $\displaystyle\sum_{n=2} \frac{1}{\ln n^{\ln n}}$ converges by comparison to the convergent p-series $\sum 1/n^2$. ●

If a series converges by the Comparison Test, then we have the following remainder estimate, which we conclude the section with.

The Comparison Test Remainder Estimate. Let $\sum a_n$ and $\sum b_n$ be series with positive terms such that $a_n \leqslant b_n$ for $n \geqslant N$. Then for $n \geqslant N$, the error in the nth partial sum of $\sum a_n$, s_n, is bounded by $b_{n+1} + b_{n+2} + \cdots$:

$$\left| s_n - \sum_{n=1}^{\infty} a_n \right| \leqslant b_{n+1} + b_{n+2} + \cdots .$$

Proof. By definition,

$$\left| s_n - \sum_{n=1}^{\infty} a_n \right| = \left| -a_{n+1} - a_{n+2} - \cdots \right|.$$

Now because the terms of $\sum a_n$ are positive, this is $a_{n+1} + a_{n+2} + \cdots$, and since we have assumed that $n \geqslant N$,

$$a_{n+1} + a_{n+2} + \cdots \leqslant b_{n+1} + b_{n+2} + \cdots,$$

proving the estimate. □

Example 11. How many terms are needed to approximate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + n}$ to within $1/10$?

Solution. We use the comparison

$$\frac{1}{2^n + n} \leqslant \left(\frac{1}{2}\right)^n$$

for all $n \geqslant 1$ to bound the error in approximating $\displaystyle\sum \frac{1}{2^n + n}$. The first partial sum may not be a good enough approximation:

$$\left| s_1 - \sum_{n=1}^{\infty} \frac{1}{2^n + n} \right| \leqslant \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = \frac{\left(\frac{1}{2}\right)^2}{1 - \frac{1}{2}} = \frac{1}{2}.$$

The second and third partial sums are also not guaranteed to be as close to the true sum as required:

$$\left| s_2 - \sum_{n=1}^{\infty} \frac{1}{2^n + n} \right| \leqslant \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots = \frac{\left(\frac{1}{2}\right)^3}{1 - \frac{1}{2}} = \frac{1}{4},$$

$$\left| s_3 - \sum_{n=1}^{\infty} \frac{1}{2^n + n} \right| \leqslant \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \cdots = \frac{\left(\frac{1}{2}\right)^4}{1 - \frac{1}{2}} = \frac{1}{8},$$

but the fourth partial sum is within $1/10$:

$$\left| s_4 - \sum_{n=1}^{\infty} \frac{1}{2^n + n} \right| \leqslant \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \cdots = \frac{\left(\frac{1}{2}\right)^5}{1 - \frac{1}{2}} = \frac{1}{16}.$$

Therefore the answer is that 4 terms will certainly approximate the series within $1/10$. ●

EXERCISES FOR SECTION 2.5

In Exercises 1–4, assume that $\sum a_n$ and $\sum b_n$ are both series with positive terms.

1. If $a_n \leqslant b_n$ for sufficiently large n and $\sum b_n$ is convergent, what can you say about $\sum a_n$?

2. If $a_n \leqslant b_n$ for sufficiently large n and $\sum b_n$ is divergent, what can you say about $\sum a_n$?

3. If $a_n \geqslant b_n$ for sufficiently large n and $\sum b_n$ is convergent, what can you say about $\sum a_n$?

4. If $a_n \geqslant b_n$ for sufficiently large n and $\sum b_n$ is divergent, what can you say about $\sum a_n$?

Determine if the series in Exercises 5–18 converge or diverge.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n-3}{n^3}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+7}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+4}}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{\sqrt[3]{n^5+8}}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{9}{3^n+1}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{\sqrt{n^3+1}}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3-1}{n^5+1}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{2-\sin(n)}{n^{5/4}}$

15. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n+5^n}$

17. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{3n^2}\right)^2$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$

Suppose that $\sum a_n$ is a convergent series with positive terms. Determine whether the series listed in Exercises 19–22 necessarily converge. If a series doesn't necessarily converge, give an example of a convergent series $\sum a_n$ with positive terms for which it diverges. It may be helpful to remember that there are only finitely many values of a_n at least 1, so these have no affect on the convergence of the series.

19. $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n} a_n$

21. $\displaystyle\sum_{n=1}^{\infty} n a_n$

22. $\displaystyle\sum_{n=1}^{\infty} a_n \sin n$

23. $\displaystyle\sum_{n=1}^{\infty} a_n^2$

24. $\displaystyle\sum_{n=1}^{\infty} \sqrt{a_n}$

In Exercises 25–28, use reindexing like we did in Examples 5–8 to determine if the given series converge or diverge.

25. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-3}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{n^2}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 2^n - n}$

Using the Comparison Test, determine if the series in Exercises 29–38 converge or diverge.

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n + \sqrt{n+1}}}$

31. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^{n^2}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{2n(\ln n)^4}{\sqrt{n^4+4}}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n(n+1)}\sqrt{\frac{\ln n}{n}}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln \ln n)^{\ln n}}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln n)^{\ln \ln n}}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n!}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

39. Construct an example showing that the Comparison Test need not hold if $\sum a_n$ and $\sum b_n$ are not required to have positive terms.

40. If $a_n, b_n \geqslant 0$ and $\sum a_n^2$ and $\sum b_n^2$ both converge, show that the series $\sum a_n b_n$ converges.

41. Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{p-\sin n}}$$

converges for $p > 2$. What about when $p = 2$?

Another way to deal with problems like Exercises 25–28 is to apply the following test.

The Limit Comparison Test. Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$ is a finite number, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

Exercise 42 leads you through the proof of the Limit Comparison Test. After that, Exercises 43–48 present applications, while Exercises 49 and 50 extend the Limit Comparison Theorem to the case where the limit is 0 or ∞.

◆ **42.** Suppose that $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite number. Therefore there are positive numbers m and M with $m < c < M$ such that $m < \dfrac{a_n}{b_n} < M$ for all large n. Use this inequality and the Comparison Test to derivate the Limit Comparison Test.

43. Show that the series

$$\sum_{n=2}^{\infty} \frac{n^2+3}{n^4-2}$$

from Example 8 converges using the Limit Comparison Test.

44. Does $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ converge or diverge?

45. Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-2n+1}{\sqrt[5]{n^1 1+11n}}$ converge or diverge?

46. Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-2n+1}{\sqrt[5]{n^9+11n}}$ converge or diverge?

◆ **47.** Suppose that $a_n \geqslant 0$ and $a_n \to 0$. Show that $\sum \sin a_n$ converges if and only if $\sum a_n$ converges.

◆ **48.** Suppose that $0 \leqslant a_n < 1$ for all n. Prove that $\sum \arcsin a_n$ converges if and only if $\sum a_n$ converges.

◆ **49.** Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If $a_n/b_n \to 0$ and $\sum b_n$ converges, prove that $\sum a_n$ converges.

◆ **50.** Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If $a_n/b_n \to \infty$ and $\sum b_n$ diverges, prove that $\sum a_n$ diverges.

ANSWERS TO SELECTED EXERCISES, SECTION 2.5

1. $\sum a_n$ converges, by the Comparison Test
3. You cannot conclude anything
5. Converges by comparison to $\sum {}^{n}\!/\!{}_{n^3} = \sum {}^{1}\!/\!{}_{n^2}$
7. Converges by comparison to $\sum {}^{1}\!/\!{}_{\sqrt{n^3}} = \sum {}^{1}\!/\!{}_{n^{3/2}}$
9. Converges by comparison to $\sum {}^{n}\!/\!{}_{n^{5/2}} = \sum {}^{1}\!/\!{}_{n^{3/2}}$
11. Converges by comparison to $\dfrac{\pi}{2}\sum \dfrac{1}{n^{3/2}}$
13. Converges by comparison to $\sum {}^{1}\!/\!{}_{n^2}$
15. Diverges by comparison to $\sum {}^{1}\!/\!{}_{\sqrt{n}}$
17. Converges by comparison to $\sum {}^{1}\!/\!{}_{9n^2}$
19. Converges by the Comparison Test: $a_n \leqslant \dfrac{a_n}{n}$
21. Need not converge, consider taking $a_n = {}^{1}\!/\!{}_{n^2}$
23. Since $\sum a_n$ converges, $a_n \leqslant 1$ for sufficiently large n. For these values of n, $a_n^2 \leqslant a_n$, so $\sum a_n^2$ converges by the Comparison Test