Math 8 Midterm Exam I Answers, Winter 2004

1.) Short Answer

(i) $\lim_{n\to\infty} \frac{n^3-3\sqrt{n}+5\cos(n^3)}{n^2-1-2n^3} = -\frac{1}{2}$

(ii) The radius of convergence of the power series $\sum_{n=5}^{\infty} \frac{(2x+5)^n}{n^2 3^n}$ is $R = \frac{3}{2}$.

(iii) $-3 + \frac{5}{3} - \frac{25}{9} + \frac{125}{27} - \cdots =$ divergent series since it is a geometric series with $r = -\frac{5}{3}$ and $|r| = \frac{5}{3} > 1$.

(iv) The general solution of $y' = y + \frac{1}{y}$, $y > 0$, is $y = \sqrt{Ce^{2x} - 1}$.

(v) A mass of 2 kg is attached to a horizontal spring that has a natural length of 3 meters. A force of 6 N is required to stretch the spring to a length of 3.5 meters. The damping constant is 14. If $y(t)$ is the disaplcement of the mass from equilibrium at time t, then $y(t)$ satisfies what differential equation?

Answer: $y'' + 7y' + 6y = 0$.

2.) True or False

(i) True: The modulus of $3-4i$ is the real number 5.
(ii) False: The complex conjugate of $3-4i$ is the complex number $-3+4i$.
(iii) True: The general solution of the ODE $6y'' = -4y$ has the general form $y = C_1\cos(\beta x) + C_2\sin(\beta x)$.
(iv) True: The differential equation $x\frac{dy}{dx} = xy^2 - 5y^2$ is separable.
(v) False: The differential equation $3y'' - 8y' + 7y - 4x = 0$ is a second-order, constant coefficient, linear, homogeneous differential equation.
(vi) False: Let $a_n = n!/n^{100}$. Then the sequence $\{a_n\}_{n=1}^{\infty}$ converges to zero.
(vii) False: The imaginary part of the complex number $z = (1-2i)(1+3i)$ is -6.
(viii) False: If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges to a limit.
(ix) True: If $\lim_{k\to\infty} |a_{k+1}/a_k| = 0.5$ then $\sum_{n=1}^{\infty} a_n$ converges.
(x) True: The Maclaurin series for $\cos(7x^2)$ converges for all real values of x.

3.) Multiple Choice

(i) A square root of the complex number $w = -4i$ can be written in the form

$$z = 2(\cos(\theta) + i\sin(\theta))$$

where the angle θ is given by:

C. $\theta = \frac{3\pi}{4}$

(ii) The function $y = c_1e^{-t} + c_2e^{4t}$, where c_1 and c_2 are constants, is a general solution to which differential equation?

B. $y'' - 3y' - 4y = 0$

(iii) The Taylor polynomial of degree 3 centered at $a = 0$ for the function $f(x) = xe^{2x}$ is:

B. $x + 2x^2 + 2x^3$

(iv) The direction field for the differential equation $y' = y/x$ is represented by which diagram?

Diagram D:

Problem 4.) Consider the following first-order linear differential equation:

$$4y' - 12y - 60x = 0$$

(i) Write this equation in standard form $\frac{dy}{dx} + P(x)y = Q(x)$.

Answer: $\frac{dy}{dx} - 3y = 15x$. $P(x) = -3$ and $Q(x) = 15x$.

(ii) The differential equation in part (i.) becomes easier if we multiply both sides by what function $I(x)$?

Answer: $I(x) = e^{-3x}$.

(iii) Find the general solution to the above differential equation.

Answer: $y = -5(x + \frac{1}{3}) + Ce^{3x}$.

(iv) Find the unique solution that passes through the point $(-\frac{1}{3}, \frac{4}{e})$.

Answer: $C = 4$ so $y = -5(x + \frac{1}{3}) + 4e^{3x}$.

Problem 5.) A tank initially contains $1,000$ liters of pure water. Brine containing 0.05 kilograms of salt per liter enters through one pipe at 5 liters per minute. Brine containing 0.035 kilograms of salt per liter enters through a second pipe at a rate of 10 liters/min. Water is draining through a hose in the bottom at a rate of 15 liters/min. The tank is kept thoroughly mixed at all times.

Let $x(t)$ be the amount of salt (in kg) present in the tank after t minutes.

(i) What differential equation describes how the salt content of the tank changes wrt time?

Answer: $\frac{dx}{dt} = \frac{600-15x}{1000}$

(ii) What is the general solution to the differential equation in part (i)?

Answer: $x(t) = 40 - Ce^{-15t/1000}$

(iii) What additional piece of information allows one to **uniquely** determine the amount of salt present at time t?

Answer: $x(0) = 0$.

(iv) How much salt is present in the tank at time t?

Answer: $C = 40 \implies x(t) = 40(1 - e^{-15t/1000})$.

(v) After one hour?

Answer: $x(60) = 40(1 - e^{-0.9})$.

Problem 5.) Let $f(x) = \sin(x)$.

(i) Find the Taylor polynomial $T_2(x)$ of degree 2 for f centered at $a = \frac{\pi}{3}$.

Answer: $T_2(x) = \frac{\sqrt{3}}{2} + \frac{1}{2}(x - \frac{\pi}{3}) - \frac{\sqrt{3}}{4}(x - \frac{\pi}{3})^2$.

(ii) Use T_2 to estimate $\sin(\frac{\pi}{3} + \frac{1}{2})$. Express your answer as a fraction $\frac{a}{b}$.

Answer: $\sin(\frac{\pi}{3} + \frac{1}{2}) \approx T_2(\frac{\pi}{3} + \frac{1}{2}) = \frac{7\sqrt{3}+4}{16}$.

(iii) Find an upper bound for the absolute error in approximating $\sin(\frac{\pi}{3} + \frac{1}{2})$ with $T_2(\frac{\pi}{3} + \frac{1}{2})$.

$f^{(3)}(x) = -\cos(x) \implies |f^{(3)}(x)| = |\cos(x)| \leq 1 = M$.

$|f(x) - T_2(x)| = |R_2(x)| \leq \frac{M|x-\frac{\pi}{3}|^3}{3!} = \frac{|x-\frac{\pi}{3}|^3}{6}$

When $x = \frac{\pi}{3} + \frac{1}{2}$ we get $|R_2(\frac{\pi}{3} + \frac{1}{2})| \leq \frac{(1/2)^3}{6} = \frac{1}{48} = E$.

Answer: $E = \frac{1}{48}$ is an upper bound for the absolute error.