Math 8 Practice Exam Problems

Disclaimer: These problems primarily from the material since the last exam. You should also consult your previous exams and the previous practive exam problems.

1. Find the general solution to the differential equation $\frac{dy}{dx} + \frac{y}{x \ln x} = x$.

ANS: This is a first order linear differential equation with integrating factor equal to $e^{\int dx/(x \ln x)}$. Now $\int \frac{dx}{x \ln x} = \ln(\ln x) + C$ using the substitution $u = \ln x$, so $e^{\int dx/(x \ln x)} = e^{\ln(\ln x)} = \ln x$. Multiplying both sides of the differential equation by this factor yields $\frac{d}{dx}(y \ln x) = x \ln x$. We integrate (the right hand side by parts): $y \ln x = \int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \frac{x^2}{4} + C$, hence $y = \frac{1}{2}x^2 - \frac{x^2}{4 \ln x} + \frac{C}{\ln x}$.

2. Find the equation of the tangent plane to the level surface of $f(x, y, z) = ye^{-x^2} \sin z$ at $(0, 1, \pi/3)$.

ANS: We need only compute the gradient of f at $(0, 1, \pi/3)$.
$\nabla f(x, y, z) = \langle -2xye^{-x^2} \sin z, e^{-x^2} \sin z, ye^{-x^2} \cos z\rangle$, so $\nabla f(0, 1, \pi/3) = \langle 0, \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle$, and the equation of the plane is $0(x - 0) + \frac{\sqrt{3}}{2}(y - 1) + \frac{1}{2}(z - \pi/3) = 0$, or more simply, $\sqrt{3}y + z = \sqrt{3} + \pi/3$.

3. Suppose that $z = f(x, y)$ is a smooth real-valued function of two variables, and that $\frac{\partial f}{\partial x}(1, 1) = 3$ and $\frac{\partial f}{\partial y}(1, 1) = -1$. If $x = s^2$ and $y = s^3$, we may then view z as a function of the single variable s. The value of $\frac{dz}{ds}$ at $s = 1$ is

ANS: By the chain rule, $\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds} = \frac{\partial z}{\partial x}2s + \frac{\partial z}{\partial y}3s^2$. When $s = 1$, $x = y = 1$, so we have $\left.\frac{dz}{ds}\right|_{s=1}$ is 3(2) + (-1)(3) = 3.

4. Find an equation of the curve $y = f(x)$ that passes through the point $(1, 1)$ and intersects all level curves of the function $g(x, y) = x^4 + y^2$ at right angles.

ANS: $\nabla g(x, y) = \langle 4x^3, 2y\rangle$ is always orthogonal to the level curves of $g(x, y)$, so we want the tangent line to $y = f(x)$ to be parallel to the gradient at each point (x, y), that is, the slope $\frac{dy}{dx} = \frac{2y}{4x^3} = \frac{y}{2x^3}$. This is a separable differential equation: $\frac{dy}{y} = \frac{dx}{2x^3}$. Integrating, we obtain $\ln |y| = -x^{-2}/4 + C$, so $y = Ae^{-x^{-2}/4}$. Given that the curve passes through $(1, 1)$, we have $y = e^{1/4}e^{-x^{-2}/4}$.

5. A ball is placed at the point $(1, 2, 3)$ on the surface $z = y^2 - x^2$. Give the direction in the xy-plane corresponding to the direction in which the ball will start to roll. Describe the path in the xy-plane which the ball will follow.

ANS: Let $f(x, y) = y^2 - x^2$. The gradient of f points in the direction of maximum increase of f (the height), so $-\nabla f$ points in the direction of maximum decrease, and hence the direction in which the ball will roll. $\nabla f(x, y) = \langle -2x, 2y\rangle$, so $-\nabla f(1, 2) = \langle 2, -4\rangle$ indicates the direction in which the ball will start to roll. The path followed by the ball is similar to the previous problem, namely we expect $\frac{dy}{dx} = \frac{-y}{x}$. This yields $y = 2/x$ for a solution.

6. Let $f(x, y) = x^4 + y^4 + x^2 - y^2$. Find and classify all critical points of f. Use the method of Lagrange multipliers to find the largest and smallest values of f on the circle $x^2 + y^2 = 4$.

ANS: $\nabla f(x, y) = \langle 4x^3 + 2x, 4y^3 - 2y\rangle = \langle 0, 0\rangle$. From this we have $x(2x^2 + 1) = 0$ and $y(2y^2 - 1)$, so the only solutions are $x = 0$, and $y = 0, \pm 1/\sqrt{2}$.
The Hessian matrix (second derivative test) is $\begin{pmatrix} 12x^2 + 2 & 0 \\ 0 & 12y^2 - 2 \end{pmatrix}$
At the points $(0, \pm 1/\sqrt{2})$, the determinant is positive indicating local extrema which, since $f_{11}(0, \pm 1/\sqrt{2}) > 0$, are local minima. At $(0, 0)$, the Hessian has negative determinant, indicating the origin is a saddle point.
For Lagrange multipliers, let $g(x, y) = x^2 + y^2$. We need to solve the system given by $\nabla f = \lambda \nabla g$ and $g(x, y) = 4$. This gives

$$\begin{aligned} 4x^3 + 2x &= \lambda 2x \\ 4y^3 - 2y &= \lambda 2y \\ x^2 + y^2 &= 4 \end{aligned}$$

We have to exercise a little caution here. We would like to say that $\lambda = 2x^2 + 1 = 2y^2 - 1$, but this is true only if x and y are not zero. Since zero is a possible value for x and y, we must make three cases: $x = 0$ yielding the two points $(0, \pm 2)$; $y = 0$ yielding the two points $(\pm 2, 0)$, and the "desired" case where $2x^2 + 1 = 2y^2 - 1$. In this last case, we have $x^2 - y^2 = -1$ and $x^2 + y^2 = 4$ which yields (adding the two equations) $2x^2 = 3$. $x^2 = 3/2$ implies $y^2 = 5/2$ which provides 4 points $(\pm\sqrt{3/2}, \pm\sqrt{5/2})$.

Testing all points we see that $f(0,\pm 2) = 12$, $f(\pm 2, 0) = 20$, and $f(\pm\sqrt{3/2}, \pm\sqrt{5/2}) = 15/2$. Thus the maximum value is 20 which occurs at $(\pm 2, 0)$, and the minimum value is 15/2 which occurs at $(\pm\sqrt{3/2}, \pm\sqrt{5/2})$.

7. Consider $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$. Does the limit exist? Why or why not?

ANS: No the limit does not exist. Let $f(x,y) = (x^2 - y^2)/(x^2 + y^2)$. Along the line $y = x$, we have $\lim_{x\to 0} f(x,x) = 0$. But along the x-axis, $\lim_{x\to 0} f(x,0) = 1$. Since these limits do not agree, the limit does not exist.

8. The temperature at the point (x,y,z) is given by $T(x,y,z) = xy^2z$. Find the direction of maximum increase in temperature at the point $(1,-2,3)$. If you move so that your velocity as you pass through the point $(1,-2,3)$ is $(1,2,2)$, then what is the rate of temperature increase as you pass through $(1,-2,3)$?

ANS: The maximum temperature increase occurs in the direction of the gradient $\nabla T(1,-2,3) = (12,-12,4)$. Thus the direction is $\mathbf{u} = \frac{1}{|\nabla T(1,-2,3)|}\nabla T(1,-2,3) = \frac{1}{\sqrt{19}}(3,-3,1)$. If your velocity is $\mathbf{v} = (1,2,2)$, then the rate of temperature increase is $D_{\mathbf{v}}T(1,-2,3) = \mathbf{v} \bullet \nabla T(1,-2,3) = (1,2,2) \cdot (12,-12,4) = -8$.