Homework 3 due Sep. 29

1.
$$\int_{\pi/4}^{\pi/2} x\csc^2 x\,dx = x\cdot(-\cot x)\Big]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2}(-\cot x)\,dx \quad \begin{cases} u = x & dv = \csc^2 x\,dx \\ du = dx & v = -\cot x \end{cases}$$

$$= \left(-x\cdot\cot x - (-\ln|\sin x|)\right)\Big]_{\pi/4}^{\pi/2}$$

$$= \left(-\tfrac{\pi}{2}\cot(\tfrac{\pi}{2}) + \ln|\sin(\tfrac{\pi}{2})|\right) - \left(-\tfrac{\pi}{4}\cot(\tfrac{\pi}{4}) + \ln|\sin(\tfrac{\pi}{4})|\right)$$

$$= \left(-\tfrac{\pi}{2}\cdot 0 + \ln(1)\right) - \left(-\tfrac{\pi}{4}\cdot 1 + \ln(\tfrac{1}{\sqrt{2}})\right)$$

$$= 0 - \left(-\tfrac{\pi}{4} + \ln(\sqrt{2}^{-1})\right)$$

$$= \tfrac{\pi}{4} - (-\ln(\sqrt{2}))$$

$$= \tfrac{\pi}{4} + \ln(\sqrt{2}).$$

Note $\ln(\sqrt{2}) = \ln(2^{1/2}) = \frac{1}{2}\cdot\ln 2$, so we may rewrite this as

$$\int_{\pi/4}^{\pi/2} x\csc^2 x\,dx = \frac{\pi}{4} + \frac{1}{2}\ln 2.$$

2. We follow strategy (a.) on p. 520:

$$\int \sin^6 x\cos^3 x\,dx = \int \sin^6 x\cdot(1-\sin^2 x)\cos x\,dx$$

Substitute $u = \sin x$, $du = \cos x\,dx$:

$$= \int u^6(1-u^2)\,du$$

$$= \int (u^6 - u^8)\,du$$

$$= \tfrac{1}{7}u^7 - \tfrac{1}{9}u^9 + C$$

$$= \tfrac{1}{7}\sin^7 x - \tfrac{1}{9}\sin^9 x + C$$

3. This time we use strategy (b.):

$$\int \sin^3(mx)dx = \int (1-\cos^2(mx))\sin(mx)dx$$

Substitute $u = \cos(mx)$

$du = -m\sin(mx)$

so $$\int \sin^3(mx)dx = \frac{1}{(-m)}\int(1-u^2)du$$

$$= -\frac{1}{m}\left(u-\frac{1}{3}u^3\right)+C$$

$$= -\frac{1}{m}\cos(mx)+\frac{1}{3m}\cos^3(mx)+C$$

4. Integrate by parts

$u = x \qquad dv = \cos^2 x\,dx$

$du = dx \qquad v = \frac{1}{2}x+\frac{1}{4}\sin(2x)$

so $$\int x\cos^2 x\,dx = x\cdot\left(\frac{1}{2}x+\frac{1}{4}\sin(2x)\right)-\int\left(\frac{1}{2}x+\frac{1}{4}\sin(2x)\right)dx$$

$$= \frac{1}{2}x^2+\frac{x}{4}\sin(2x)-\left(\frac{1}{4}x^2-\frac{1}{4}\cdot\frac{1}{2}\cos(2x)\right)+C$$

$$= \frac{1}{4}x^2+\frac{x}{4}\sin(2x)+\frac{1}{8}\cos(2x)+C$$

5. Follow strategy (a.) on p.522:

$$\int_0^{\pi/4} \sec^4\theta \tan^4\theta \, d\theta = \int_0^{\pi/4} (1+\tan^2\theta)\tan^4\theta \cdot \sec^2\theta \, d\theta$$

Substitute $u = \tan\theta$

$du = \sec^2\theta \, d\theta$

so $$\int_0^{\frac{\pi}{4}} \sec^4\theta \tan^4\theta \, d\theta = \int_{\tan(0)}^{\tan(\frac{\pi}{4})} (1+u^2)\cdot u^4 \, du$$

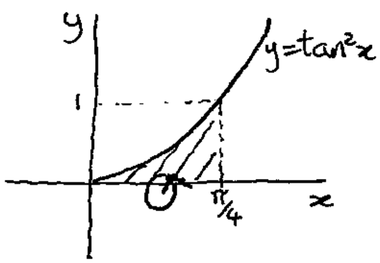
$$= \int_0^1 (u^4+u^6)\,du$$

$$= \left(\tfrac{1}{5}u^5 + \tfrac{1}{7}u^7\right)\Big|_0^1$$

$$= \tfrac{1}{5} + \tfrac{1}{7}$$

$$= \frac{12}{35}$$

6.



$$V = \int_0^{\pi/4} \pi y^2 \, dx = \pi \cdot \int_0^{\pi/4} \tan^4 x \, dx$$

$$= \pi \cdot \int_0^{\pi/4} \tan^2 x (\sec^2 x - 1)\,dx$$

$$= \pi \cdot \int_0^{\pi/4} \tan^2 x \sec^2 x \, dx - \pi \int_0^{\pi/4} \tan^2 x \, dx. \quad (*)$$

Calculate the two integrals separately:

for the first, substitute $u = \tan x$ following strategy (a.)

$du = \sec^2 x\, dx$

so $$\int_0^{\pi/4} \tan^2 x \sec^2 x\, dx = \int_{\tan(0)}^{\tan(\frac{\pi}{4})} u^2\, du = \frac{1}{3} u^3 \Big]_0^1 = \frac{1}{3}$$

while for the second, we have

$$\int_0^{\pi/4} \tan^2 x\, dx = \int_0^{\pi/4} (\sec^2 x - 1)\, dx = (\tan x - x)\Big]_0^{\pi/4} = 1 - \frac{\pi}{4}$$

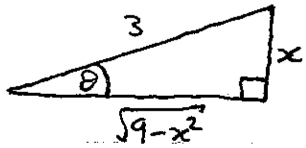
Substituting into $(*)$, we get

$$V = \pi \cdot \frac{1}{3} - \pi \cdot \left(1 - \frac{\pi}{4}\right)$$

$$= \frac{\pi^2}{4} - \frac{2}{3}\pi$$

7. Substitute $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$



$\sin\theta = \frac{x}{3}$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$$\int x^3 \sqrt{9-x^2}\, dx = \int (3\sin\theta)^3 \cdot \sqrt{9-(3\sin\theta)^2} \cdot 3\cos\theta\, d\theta$$

$$= \int 3^3 \cdot \sin^3\theta \cdot \sqrt{9} \cdot \sqrt{1-\sin^2\theta} \cdot 3\cos\theta\, d\theta$$

$$= 3^5 \int \sin^3\theta \sqrt{\cos^2\theta} \cos\theta\, d\theta$$

$$= 3^5 \int \sin^3\theta \cos^2\theta\, d\theta$$

$$= 3^5 \int (1-\cos^2\theta) \cdot \cos^2\theta \sin\theta\, d\theta$$

Substitute $u = \cos\theta$

$du = -\sin\theta \, d\theta$

$$\therefore \int x^3\sqrt{9-x^2}\,dx = 3^5\int(1-u^2)\cdot u^2\cdot(-du)$$

$$= 3^5\int(u^4-u^2)du$$

$$= 3^5\left(\tfrac{1}{5}u^5 - \tfrac{1}{3}u^3\right) + C$$

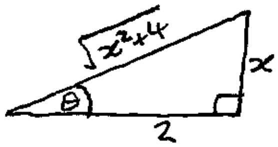
$$= 3^5\left(\tfrac{1}{5}\cos^5\theta - \tfrac{1}{3}\cos^3\theta\right) + C$$

$$= 3^5\left(\tfrac{1}{5}\cdot\left(\tfrac{\sqrt{9-x^2}}{3}\right)^5 - \tfrac{1}{3}\cdot\left(\tfrac{\sqrt{9-x^2}}{3}\right)^3\right) + C$$

$$= \tfrac{1}{5}(9-x^2)^{5/2} - 3(9-x^2)^{3/2} + C$$

8. Substitute $x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$



$$\therefore \int_0^1 x\sqrt{x^2+4}\,dx = \int_{x=0}^{x=1}(2\tan\theta)\cdot\sqrt{(2\tan\theta)^2+4}\cdot 2\sec^2\theta\,d\theta$$

$$= 4\cdot\sqrt{4}\int_{x=0}^{x=1}\tan\theta\cdot\sqrt{\tan^2\theta+1}\cdot\sec^2\theta\,d\theta$$

$$= 8\int_{x=0}^{x=1}\tan\theta\cdot\sec\theta\cdot\sec^2\theta\,d\theta$$

Substitute $u = \sec\theta$ using strategy (b.) on p.522

$du = \sec\theta\cdot\tan\theta\,d\theta$

so $\displaystyle\int_0^1 x\sqrt{x^2+4}\,dx = 8\int_{x=0}^{x=1}u^2\,du = 8\left(\tfrac{1}{3}u^3\right)\Big]_{x=0}^{x=1}$

Now $u = \sec\theta = \dfrac{1}{\cos\theta} = \dfrac{\sqrt{x^2+4}}{2}$

$$\therefore \int_0^1 x\sqrt{x^2+4}\,dx = \frac{8}{3}u^3\Big]_{u=\frac{\sqrt{4}}{2}=1}^{u=\frac{\sqrt{5}}{2}}$$

$$= \frac{8}{3}\cdot\left(\frac{\sqrt{5}}{2}\right)^3 - \frac{8}{3}$$

$$= \frac{1}{3}\left(5\sqrt{5}-8\right)$$

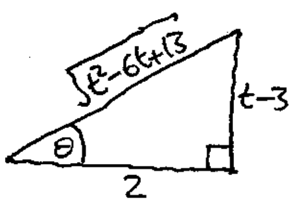
9. $t^2-6t+13 = (t^2-6t+9)+4 = (t-3)^2+4$

so $$\int \frac{dt}{\sqrt{t^2-6t+13}} = \int \frac{dt}{\sqrt{(t-3)^2+4}}$$

Substitute $t-3 = 2\tan\theta$

$dt = 2\sec^2\theta\, d\theta$

$$\therefore \int \frac{dt}{\sqrt{t^2-6t+13}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{(2\tan\theta)^2+4}} = \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta}$$



$$= \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

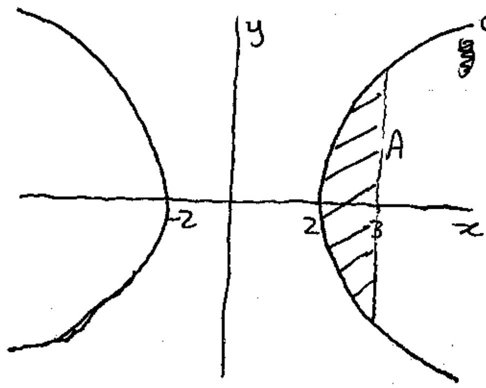
$$= \ln\left|\frac{\sqrt{t^2-6t+13}}{2} + \frac{t-3}{2}\right| + C$$

$$= \ln\left|\sqrt{t^2-6t+13} + t - 3\right| - \ln 2 + C$$

$$= \ln\left|\sqrt{t^2-6t+13} + t - 3\right| + C'$$

where C' is a new constant, $C' = C - \ln 2$.

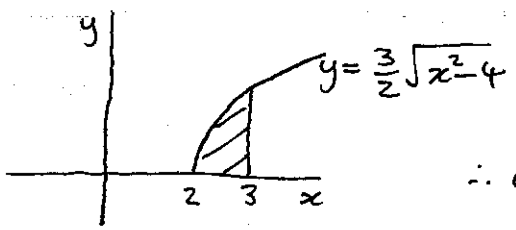
10.



Rearranging,

$$9x^2-36=4y^2$$
$$\pm\sqrt{\tfrac{9}{4}x^2-9}=y$$
$$\pm\tfrac{3}{2}\sqrt{x^2-4}=y$$

The region A (shaded) is the one whose area we want. This region is symmetric in the x-axis, i.e. there is exactly as much above the x-axis as there is below it, so we may find the area above the x-axis and double it to get the answer:



$$\therefore \text{area}(A)=2\int_2^3 \frac{3}{2}\sqrt{x^2-4}\,dx = 3\int_2^3\sqrt{x^2-4}\,dx$$

Substitute $x=2\sec\theta$:

$$dx=2\sec\theta\tan\theta\,d\theta$$

$$\therefore \text{area}(A)=3\int_{x=2}^{x=3}2\sqrt{\sec^2\theta-1}\cdot 2\sec\theta\tan\theta\,d\theta$$

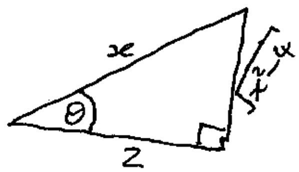
$$=12\int_{x=2}^{x=3}\sec\theta\tan^2\theta\,d\theta$$

$$=12\int_{x=2}^{x=3}\sec\theta(\sec^2\theta-1)\,d\theta$$

$$\text{area}(A) = 12\int_{x=2}^{x=3} (\sec^3\theta - \sec\theta)\,d\theta$$

$$= 12\left(\tfrac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|) - \ln|\sec\theta+\tan\theta|\right)\Big]_{x=2}^{x=3}$$

by Example 8 on p.523.



Now $\sec\theta = \frac{x}{2}$

$\tan\theta = \frac{\sqrt{x^2-4}}{2}$

so when $x=3$, $\sec\theta = \frac{3}{2}$ and $\tan\theta = \frac{\sqrt{9-4}}{2} = \frac{\sqrt{5}}{2}$

when $x=2$, $\sec\theta = 1$ and $\tan\theta = 0$

$$\therefore \text{area}(A) = 12\left(\tfrac{1}{2}\left(\tfrac{3}{2}\cdot\tfrac{\sqrt{5}}{2} + \ln\left|\tfrac{3}{2}+\tfrac{\sqrt{5}}{2}\right|\right) - \ln\left|\tfrac{3}{2}+\tfrac{\sqrt{5}}{2}\right|\right) - 12\cdot\left(\tfrac{1}{2}(0+\ln(1)) - \ln(1)\right)$$

$$= 6\left(\frac{3\sqrt{5}}{4} - \ln\left(\frac{3+\sqrt{5}}{2}\right)\right)$$

Since $\frac{3+\sqrt{5}}{2} > 0$, we may remove the absolute value signs.

$$= \frac{9\sqrt{5}}{2} - 6\ln\left(\frac{3+\sqrt{5}}{2}\right)$$