LECTURE OUTLINE
Partial Fractions and Improper Integral

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Math 8

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Goals

SERIOUS Algebra "Review"

Partial Fraction Decomposition

Improper Integrals

Polynomials

$P(x)$ is *polynomial* with real coefficients provided

$$P(x) = \sum_{k=0}^{n} a_k x^k$$

where the a_k are real numbers. $P(x)$ is said to have degree n provided $a_n \neq 0$.

Division

Division Theorem: If $P(x)$ and $Q(x)$ are nonzero polynomials with real coefficients, then there exist polynomials $S(x)$ and $R(x)$ such that

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

with the degree of $R(x)$ less than the degree of $Q(x)$.

Example 1

Let

$$P(x) = x^6 - 3x^5 + 5x^4 - 4x^3 + 3x^2 - 4x + 4$$

and

$$Q(x) = x^4 - 4x^3 + 8x^2 - 8x + 3.$$

Divide $P(x)$ by $Q(x)$.

Factoring a Polynomial

Factorization Theorem: Every polynomial $Q(x)$ with real coefficients, is equal to

$$c(x-r_1)^{N_1}....(x-r_n)^{N_n}((x-a_1)^2+b_1^2)^{M_1}....((x-a_m)^2+b_m^2)^{M_m}$$

for some real numbers c, r_i, a_i and b_i such that $(x-r_i) \neq (x-r_j)$ when $i \neq j$ and $((x-a_i)^2+b_i^2) \neq ((x-a_j)^2+b_j^2)$ when $i \neq j$.

Example 2

Factor the polynomial

$$Q(x) = x^4 - 4x^3 + 8x^2 - 8x + 3.$$

Holy Moly! what a mess!

Partial Fraction Decomposition: A rational function $R(x)/Q(x)$ with $Q(x)$ equal to

$$(x-r_1)^{N_1}....(x-r_n)^{N_n}((x-a_1)^2+b_1^2)^{M_1}....((x-a_m)^2+b_m^2)^{M_m}$$

can be expressed as

$$\frac{R(x)}{Q(x)} = \sum_{i=1}^{n}\left(\sum_{k=1}^{N_i}\frac{A_{k,i}}{(x-r_i)^k}\right) + \sum_{i=1}^{m}\left(\sum_{k=1}^{M_i}\frac{B_{k,i}x + C_{k,i}}{((x-a_i)^2+b_i^2)^k}\right)$$

for some real numbers $A_{k,i}$, $B_{k,i}$ and $C_{k,i}$.

Example 3

Find the partial fraction decomposition of

$$\frac{x+1}{x^4 - 4x^3 + 8x^2 - 8x + 3}.$$

Using this Decomposition

This reduces integrating $P(x)/Q(x)$ to integrating functions in the form x^n,

$$\frac{1}{(x-r)^n},$$

or

$$\frac{1}{((x-a)^2+b^2)^m}.$$

You can do these integrals!

Example 4

Find an anti-derivative of

$$\frac{x^6 - 3x^5 + 5x^4 - 4x^3 + 3x^2 - 4x + 4}{x^4 - 4x^3 + 8x^2 - 8x + 3}.$$

Improper Integral

If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided this limit exists (as a finite number). We say the integral is *convergent* if the limit exist and *divergent* otherwise.

Example 5

Find

$$\int_1^\infty \frac{1}{x^p} dx.$$

Improper Integral

If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b} \int_a^t f(x)dx$$

provided this limit exists (as a finite number).

Example 6

Find

$$\int_0^1 \frac{1}{(1-x)^p} dx.$$

Example 7

Find

$$\int_3^7 \frac{1}{\sqrt{x-3}}dx.$$