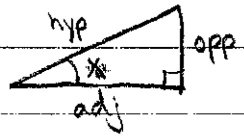
Homework #2 - Sept. 27, 2004



#1 Find the exact value of $\csc^{-1} 2$

We want to find $x$ such that $\csc x = 2$ and $x \in (0, \pi/2] \cup (\pi, 3\pi/2]$

$\csc x = \dfrac{\text{hyp}}{\text{opp}}$

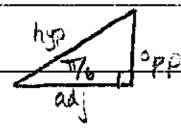



When $x = \pi/6$, $\dfrac{\text{hyp}}{\text{opp}} = 2$, so $\csc \pi/6 = 2$

and $\pi/6 \in (0, \pi/2] \cup (\pi, 3\pi/2]$. So $\boxed{\csc^{-1} 2 = \pi/6}$

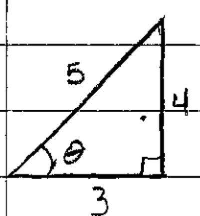
#2 Find the exact value of $\cos(\arcsin \frac{1}{2})$

$\arcsin \frac{1}{2} = \pi/6$ since $\sin \pi/6 = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{1}{2}$




Then $\cos(\arcsin \frac{1}{2}) = \cos \dfrac{\pi}{6} = \boxed{\dfrac{\sqrt{3}}{2}}$

#3 Find the exact value of $\csc(\arccos \frac{3}{5})$.




Let $\theta = \arccos \frac{3}{5}$ So, adj $= 3$, hyp $= 5$.

opp $= 4$, since $3^2 + 4^2 = 5^2$

Then $\csc \theta = \dfrac{\text{hyp}}{\text{opp}} = \boxed{\dfrac{5}{4} = \csc(\arccos \frac{3}{5})}$

#4 Prove that $\frac{d}{dx}(\cot^{-1} x) = \dfrac{-1}{1+x^2}$

Let $y = \cot^{-1} x$. Then, $x = \cot y$.

Taking the derivative of $y$ with respect to $x$ on both sides,

we have $1 = -\csc^2 y \dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{-1}{\csc^2 y}$

Since $1 + \cot^2 y = \csc^2 y$, $\dfrac{dy}{dx} = \dfrac{-1}{1+\cot^2 y} = \dfrac{-1}{1+x^2}$ since $x = \cot y$

$\blacksquare$

#5 Find the derivative of $f(x) = x \ln(\arctan x)$.

We will use the product rule

$$\frac{d}{dx}[g(x)h(x)] = g(x)\frac{d}{dx}[h(x)] + h(x)\frac{d}{dx}[g(x)]$$

Here, $g(x) = x$ and $h(x) = \ln(\arctan x)$. We will also use the chain rule to find the derivative of $h(x) = \ln(\arctan x)$.

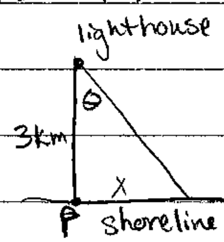
$$f' = x \cdot \frac{d}{dx}[\ln(\arctan x)] + \ln(\arctan x)\frac{d}{dx}[x]$$

$$= x \cdot \frac{1}{\arctan x} \cdot \frac{1}{1+x^2} + \ln(\arctan x) \cdot 1$$

$$= \boxed{\frac{x}{(1+x^2)\arctan x} + \ln(\arctan x)}$$

#6 A lighthouse is located on a small island, 3 km away from the nearest point P on a straight shoreline, and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

First, draw a picture




Since the light makes 4 rev/min, $\frac{d\theta}{dt} = \frac{4 \text{ rev}}{\text{min}} \cdot \frac{2\pi \text{ rad}}{\text{rev.}} \cdot \frac{60 \text{ min}}{\text{hr}} = \frac{8\pi \cdot 60 \text{ ra}}{\text{hr}}$

We know $\tan\theta = \frac{x}{3}$, so $\theta = \tan^{-1}\left(\frac{x}{3}\right)$. We want to find $\frac{dx}{dt}$ to find the speed of light when $x = 1$.

$\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt}$. We know $\frac{d\theta}{dt} = 8\pi \cdot 60$, and we also know $\frac{d\theta}{dx} = \frac{1/3}{1+(x/3)^2}$ So, $8\pi \cdot 60 = \frac{1/3}{1+(x/3)^2} \frac{dx}{dt}$

and then $\frac{dx}{dt} = 8\pi \cdot 60 \cdot 3\left(1+(x/3)^2\right)$.

Let $x=1$, and we have $\frac{dx}{dt} = 8\pi \cdot 60 \cdot 3\left(1+1/9\right) = \boxed{1600\pi \text{ km/hr}}$

#7 Evalute $\int t \sin 2t \, dt$. Use integration by parts.

Let $u=t$ & $dv = \sin 2t$, so $du = dt$ & $v = \frac{-1}{2}\cos 2t$.

So, we have $\int t \sin 2t \, dt = t\left(\frac{-1}{2}\cos 2t\right) - \int \frac{-1}{2}\cos 2t \, dt$

$$= \frac{-t\cos 2t}{2} + \frac{1}{2}\int \cos 2t \, dt$$

$$= \frac{-t\cos 2t}{2} + \frac{1}{2}\left[\frac{1}{2}\sin 2t\right] + C = \boxed{\frac{-t\cos 2t}{2} + \frac{\sin 2t}{4} + C}$$

#8 Evalute $\int \sin^{-1} x \, dx$.

Let $u = \sin^{-1} x$ & $dv = dx$, so $du = \frac{dx}{\sqrt{1-x^2}}$ & $v = x$

So, we have $\int \sin^{-1} x \, dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$ Let $t = 1-x^2$, $dt = -2x\,dx$, $\frac{dt}{-2} = x\,dx$

Then $-\int \frac{x}{\sqrt{1-x^2}} dx = -\int \frac{1}{\sqrt{t}} \cdot \frac{dt}{-2} = \frac{1}{2}\int t^{-1/2} dt = t^{1/2} + C = \sqrt{1-x^2} + C$

Putting this all together we have,

$$\boxed{\int \sin^{-1} x \, dx = x\sin^{-1} x + \sqrt{1-x^2} + C}$$

#9 First make a substitution and then use integration by parts to evaluate $\int x^5 e^{x^2} dx$

Let $t = x^2$, then $dt = 2x\,dx \Rightarrow \frac{dt}{2} = x\,dx$.

$$\int x^5 e^{x^2} dx = \int (x^2)^2 e^{x^2} x\, dx = \frac{1}{2}\int t^2 e^t dt$$

Use integration by parts with $u = t^2$ $dv = e^t dt$
$du = 2t\,dt$ $v = e^t$

So $\frac{1}{2}\int t^2 e^t dt = \frac{1}{2}\left[t^2 e^t - \int 2t e^t dt\right]$ $(*)$

This is better, but we must use integration by parts again to solve $\int 2te^t dt$. $u = t$ $dv = e^t dt$
$du = dt$ $v = e^t$

$$\int 2te^t dt = 2\left[te^t - \int e^t dt\right] = 2te^t - 2e^t + C$$

Substituting this into $(*)$, we have

$$\frac{1}{2}\int t^2 e^t dt = \frac{1}{2}t^2 e^t - \frac{1}{2}\left[2te^t - 2e^t + C\right] = \frac{1}{2}t^2e^t - te^t + e^t + C_1$$

where $C_1 = \frac{-1}{2}C$

$$= \frac{1}{2}(t^2 - 2t + 2)e^t + C$$

Since $t = x^2$, $\boxed{\int x^5 e^{x^2} dx = \frac{1}{2}\left(x^4 - 2x^2 + 2\right)e^{x^2} + C}$

#10 Use integration by parts to prove

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$$

Let $u = (\ln x)^n \qquad dv = dx$

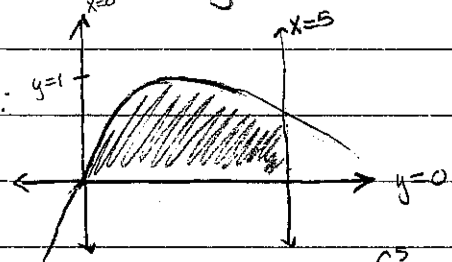
$du = n(\ln x)^{n-1}\left(\dfrac{dx}{x}\right) \qquad v = x$

So $\displaystyle\int (\ln x)^n\,dx = (\ln x)^n x - \int x\left(n(\ln x)^{n-1}\right)\frac{dx}{x}$

$\displaystyle = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$ ☒

#11 Find the area of the region bounded by $y = xe^{-0.4x}$ $y=0$ $x=5$

Draw a picture:




The area under the curve is $\displaystyle\int_0^5 x e^{-0.4x}\,dx = \text{Area}$

Integrate by parts. $u = x \quad dv = e^{-0.4x}dx$

$du = dx \quad v = -2.5e^{-0.4x}$

So, $\displaystyle \text{Area} = x\left(-2.5e^{-0.4x}\right)\Big|_0^5 - \int_0^5 -2.5e^{-0.4x}\,dx$

$\displaystyle = -2.5xe^{-0.4x}\Big|_0^5 + 2.5\int_0^5 e^{-0.4x}\,dx = -12.5e^{-2} + 0 + 2.5\left[-2.5e^{-0.4x}\right]_0^5$

$= -12.5e^{-2} - 6.25(e^{-2}-1) = \boxed{6.25 - 18.75e^{-2} \text{ -or- } \frac{25}{4} - \frac{75}{4}e^{-2}}$

Area ↗