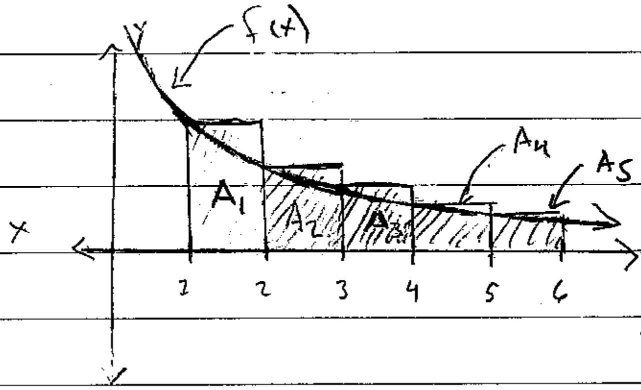
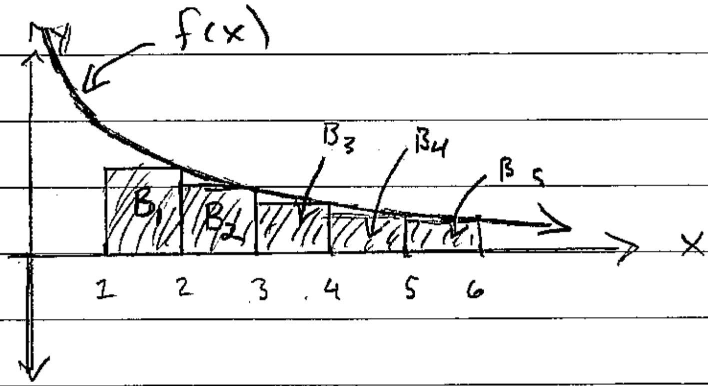
1) Suppose f is a continuous positive decreasing function for $x \geq 1$ and $a_n = f(n)$. By drawing a picture, rank the following three quantities in increasing order:

$$\int_1^6 f(x)\,dx, \quad \sum_{i=1}^{5} a_i, \quad \sum_{i=2}^{6} a_i$$





notice: $\text{area}(A_i) = 1 \cdot f(i) = a_i$

$$\sum_{i=1}^{5} \text{area}(A_i) = \sum_{i=1}^{5} a_i$$

giving, since f is decreasing

$$\int_1^6 f(x)\,dx \leq \sum_{i=1}^{5} \text{area}(A_i) = \sum_{i=1}^{5} a_i$$

notice: $\text{area}(B_i) = 1 \cdot f(i+1) = a_{i+1}$

implying $\sum_{i=1}^{5} \text{area}(B_i) = \sum_{i=1}^{5} a_{i+1} = \sum_{i=2}^{6} a_i$

giving since f is decreasing

$$\sum_{i=2}^{6} a_i = \sum_{i=1}^{5} \text{area}(B_i) \leq \int_1^6 f(x)\,dx$$

Thus $\sum_{i=2}^{6} a_i \leq \int_1^6 f(x)\,dx \leq \sum_{i=1}^{5} a_i$ ∎

Notice: $a_i \geq a_{i+1}$ since f is decreasing

implying $a_1 + a_2 + a_3 + a_4 + a_5 \geq a_2 + a_2 + a_3 + a_4 + a_5 \geq a_2 + a_3 + a_3 + a_4 + a_5$

$\geq a_2 + a_3 + a_4 + a_4 + a_5 \geq a_2 + a_3 + a_4 + a_5 + a_6$

thus we immediately have $\sum_{i=1}^{5} a_i \geq \sum_{i=2}^{6} a_i$

2) Use the integral test to determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$$

if $f(x) = \frac{1}{x^{1/4}}$ then f is decreasing, on $(0, \infty)$ and f is continuous and possitive on $(0,\infty)$ thus we can apply the integral test

$\left(\frac{\partial}{\partial x} f(x) = \frac{\partial}{\partial x} x^{-\frac{1}{4}} = -\frac{1}{4} x^{-\frac{5}{4}} \right.$ which is negative for $\left. x > 0 \right)$

$$\int_1^{\infty} \frac{dx}{x^{\frac{1}{4}}} = \int_1^{\infty} x^{-\frac{1}{4}}\, dx = \lim_{t \to \infty} \int_1^{t} x^{-\frac{1}{4}}\, dx = \lim_{t \to \infty} \left. \frac{x^{1-\frac{1}{4}}}{1-\frac{1}{4}} \right|_1^t$$

$$= \lim_{t \to \infty} \left. \frac{x^{3/4}}{3/4} \right|_1^t = \lim_{t \to \infty} \frac{4t^{3/4}}{3} - \frac{4}{3}(1)^{3/4} = \lim_{t \to \infty} \frac{4}{3} t^{3/4} - \frac{4}{3}$$

$= \infty$. Thus by the integral test $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$ diverges.

$\square$

Alternatively $\boxed{1}$ in the text gives

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \leq 1$, here $p = \frac{1}{4} < 1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{4}}}$ diverges

3) Determine whether the series is convergent or divergent.

$$1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

by $\boxed{7}$ in the text $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$

for the series in our problem $p = \frac{3}{2} > 1$

thus $1 + \frac{1}{2\sqrt{2}} + \cdots$ is convergent.

4) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

Let $f(x) = (\ln x)\, x^{-2}$ then $f'(x) = \frac{1}{x} \cdot x^{-2} + (-2x^{-3}) \ln x$

$$= \frac{1}{x^3} - \frac{2\ln x}{x^3} = \frac{1-2\ln x}{x^3}$$

for $x > e^{1/2}$ $f'(x) < 0$

thus $f(x)$ is decreasing on $(e^{1/2}, \infty)$

it is also continuous and positive on $(e^{1/2}, \infty)$ thus the integral test can be applied.

$$\int_{e^{1/2}}^{\infty} \frac{\ln x}{x^2}\, dx$$ using integration by parts with

$u = \ln x$ $du = \frac{1}{x}\, dx$

$dv = \frac{dx}{x^2} = x^{-2}\, dx$ $v = -x^{-1} = -\frac{1}{x}$

thus

$$\int_{e^{1/2}}^{\infty} \frac{\ln x}{x}\, dx = \lim_{t \to \infty} \int_{e^{1/2}}^{t} \frac{\ln x}{x}\, dx = \lim_{t \to \infty} \left[-\frac{\ln x}{x} \Big|_{e^{1/2}}^{t} - \int_{e^{1/2}}^{t} \left(-\frac{1}{x}\right) \frac{dx}{x} \right]$$

$$= \lim_{t \to \infty} \left[-\frac{\ln t}{t} - \left(-\frac{\ln e^{1/2}}{e^{1/2}} \right) + \int_{e^{1/2}}^{t} \frac{dx}{x^2} \right]$$

$$= \lim_{t \to \infty} \left[-\frac{\ln t}{t} + \frac{\frac{1}{2}}{e^{1/2}} - \frac{1}{x} \Big|_{e^{1/2}}^{t} \right] = \lim_{t \to \infty} \left[\frac{1}{2e^{1/2}} - \frac{\ln t}{t} - \frac{1}{t} - \left(-\frac{1}{e^{1/2}}\right) \right]$$

$$= \left[\frac{3}{2e^{1/2}} + \lim_{t \to \infty} \frac{-\ln t}{t} + \lim_{t \to \infty} \left(-\frac{1}{t}\right) \right] = \frac{3}{2e^{1/2}} - \lim_{t \to \infty} \frac{\ln t}{t}$$

So it remains to find $\lim_{t \to \infty} \frac{\ln t}{t}$

Using L'hopital's rule we get

$$\lim_{t \to \infty} \frac{\ln t}{t} = \lim_{t \to \infty} \frac{\frac{1}{t}}{1} = 0$$

thus $\displaystyle\int_{e^{1/2}}^{\infty} \frac{\ln x}{x}\, dx = \frac{3}{2e^{1/2}} < \infty$

Hence the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges.

5) Find the values of p for which the series is convergent

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

Let $f(x) = \dfrac{1}{x(\ln x)^p} = x^{-1}(\ln x)^{-p}$

$$f'(x) = -x^{-2}(\ln x)^{-p} + x^{-1}(-p)(\ln x)^{-p-1} \cdot \frac{1}{x}$$

$$= \frac{-1}{x^2(\ln(x))^p} - \frac{p}{x^2(\ln x)^{p+1}} = \frac{-\ln(x) - p}{x^2(\ln(x))^p}$$

is less than zero as long as

$\ln(x) + p > 0 \qquad x > e^{p}$

$f(x)$ is also continuous and positive on $[2, \infty)$

thus the integral test applies

$$\int_2^{\infty} \frac{dx}{x(\ln(x))^p} = \lim_{t\to\infty} \int_2^{t} \frac{dx}{x(\ln(x))^p} \qquad \text{let } u = \ln x \quad \text{then } du = \frac{dx}{x}$$

$$= \lim_{t\to\infty} \int_{\ln(2)}^{\ln t} \frac{dx}{u^p} = \lim_{t\to\infty} \begin{cases} \frac{1}{(1-p)} u^{-p+1} \Big|_{\ln(2)}^{\ln t} & p \neq 1 \\ \ln(u) \Big|_{\ln(2)}^{\ln t} & p = 1 \end{cases}$$

$$= \lim_{t\to\infty} \begin{cases} \frac{1}{(1-p)(\ln t)^{p-1}} - \frac{1}{(1-p)(\ln(2))^{p-1}} & p \neq 1 \\ \ln(\ln t) - \ln(\ln(2)) & p = 1 \end{cases}$$

if $p > 1$ then $\displaystyle\lim_{t\to\infty} \frac{1}{(1-p)(\ln t)^{p-1}} - \frac{1}{(1-p)(\ln(2))^{p-1}} = -\frac{1}{(1-p)(\ln(2))^{p-1}} < \infty$

if $p < 1$ then $\lim_{t \to \infty} \frac{1}{(1-p)(\ln t)^{p-1}} - \frac{1}{(1-p)(\ln(2))^{p-1}} = \infty$

if $p = 1$ then $\lim_{t \to \infty} \ln(\ln(t)) - \ln(\ln(2)) = \infty$

Thus the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$ converges for $p > 1$

6) Use (4) to show that if s_n is the nth partial sum of the harmonic series then

$$s_n \leq 1 + \ln n$$

(4) gives $a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\,dx$

thus for the harmonic series $f(x) = \frac{1}{x}$ and we have

$$s_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \sum_{i=2}^{n} \frac{1}{i} \leq 1 + \int_1^n \frac{dx}{x}$$

now $\int_1^n \frac{dx}{x} = \ln(x)\Big|_1^n = \ln(n) - \ln(1) = \ln(n)$

which implies

$$s_n \leq 1 + \int_1^n \frac{dx}{x} = 1 + \ln(n)$$

7) Suppose Σa_n and Σb_n are series with positive terms and Σb_n is known to be divergent.

(a) if $a_n > b_n$ for all n, what can you say about Σa_n? Why?

Σa_n is divergent, since if $a_n > b_n$ $a_n \geq b_n$ so by (ii) of the comparison test Σa_n is divergent

(b) if $a_n < b_n$ for all n, what can you say about Σa_n? Why.

we can not say anything about Σa_n since Σb_n is divergent.

8) Determine whether the series converges or diverges

$$\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$$

now $n - \sqrt{n} < n$ for $n > 2$

which implies

$$\frac{1}{n-\sqrt{n}} > \frac{1}{n} \quad \text{for } n > 2$$

we know from example 12.2.7

that $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges.

$\frac{1}{n} > 0$ for $n > 2$ which implies $\frac{1}{n-\sqrt{n}} > \frac{1}{n} > 0$

for $n > 2$

Hence the comparison test applies.

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges and $\frac{1}{n} < \frac{1}{n-\sqrt{n}}$

from the comparison test we know

$\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$ diverges ∎

9) Determine whether the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$$

notice for $n \geq 1$ $3n^4+1 > 0$

and $n^2-1 \geq 1-1=0$ thus all the terms of $\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$ are positive

Next $n^2-1 < n^2$ for $n \geq 1$

and $3n^4+1 > 3n^4$ for $n \geq 1$

Hence $\frac{n^2-1}{3n^4+1} < \frac{n^2}{3n^4+1} < \frac{n^2}{3n^4} = \frac{1}{3n^2}$ for $n \geq 1$

Since $\frac{n^2-1}{3n^4+1}$ are all possitive for $n \geq 1$

so are $\frac{1}{3n^2}$

Thus the comparison test applies

Now $\sum_{n=1}^{\infty} \frac{1}{3n^2} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test

because in this series $p=2>1$

Thus since $\frac{n^2-1}{3n^4+1} < \frac{1}{3n^2}$ for $n \geq 1$

by the comparison test

$\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$ converges.

10) Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$$

notice that for $n \geq 5$ $\frac{n^2-5n}{n^3+n+1} \geq 0$

if we can show

$\sum_{n=5}^{\infty} \frac{n^2-5n}{n^3+n+1}$ diverges then $\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$ diverges

notice as $n \to \infty$ the numerator is dominated by n^2 and the denominator is dominated by n^3

we will use the limit comparison test with $b_n = \frac{n^2}{n^3} = \frac{1}{n}$

$$\text{Now } \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2-5n}{n^3+n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n(n^2-5n)}{n^3+n+1}$$

$$= \lim_{n\to\infty} \frac{n^3-5n^2}{n^3+n+1} = \lim_{n\to\infty} \frac{1-\frac{5}{n}}{1+\frac{1}{n^2}+\frac{1}{n^3}} = \frac{1}{1} = 1$$

since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ the limit comparison test applies

we know that $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

thus by the limit comparison test

$\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$ must diverge as well ☒

11) For what values of p does the series $\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)}$ converge.

if $p<0$ then $\lim_{n\to\infty} \frac{1}{n^p \ln(n)} = \lim_{n\to\infty} \frac{n^{-p}}{\ln(n)}$

which is an indeterminate form so L'hopital's rule applies.

$\lim_{n\to\infty} \frac{n^{-p}}{\ln(n)} = \lim_{n\to\infty} \frac{-p\, n^{-p-1}}{\frac{1}{n}} = \lim_{n\to\infty} -p\, n^{-p} = -\infty \neq 0$

thus $\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)}$ diverges for $p<0$

for $0 \le p \le 1$ then

$n^p \ln n \le n \ln n$ for $n \ge 2$

which implies

$\frac{1}{n^p \ln n} \ge \frac{1}{n \ln n}$

Now $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ is problem 5 with $p=1$

thus $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ diverges

and since $\frac{1}{n^p \ln(n)} > 0$ for $n > 2$

the comparison test gives $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ diverges

for $p > 1$ $n^p \ln n \ge n^p$ for $n \ge 2$

which implies $\frac{1}{n^p \ln(n)} \le \frac{1}{n^p}$

$\sum_{n=2}^{\infty} \frac{1}{n^p}$ converges by the p-series test

again we know $\frac{1}{n^p \ln(n)} > 0$ for $n > 2$
thus the comparison test applies
giving
$\sum_{n=2}^{\infty} \frac{1}{n^p \ln(p)}$ converges

so $\sum_{n=2}^{\infty} \frac{1}{n^p \ln(p)}$ is convergent for $p > 1$

and divergent otherwise.