(1) Determine whether the following integral is convergent or divergent. If it is convergent, evaluate it. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ Note that there is a discontinuity at ± 1.

So $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-} \int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-} [\sin^{-1}x]_0^t = \lim_{t\to 1^-} \sin^{-1}t - \sin^{-1}0$

$= \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2} \text{ convergent}}$

(2) Determine whether the following integral is convergent or divergent. If it is convergent, evaluate it. $\int_0^1 \frac{1}{4y-1}dy$ Note that there is an infinite discontinuity at $4y-1=0$, or $y = 1/4$

So $\int_0^1 \frac{1}{4y-1}\,dy = \lim_{t\to \frac{1}{4}^-} \int_0^t \frac{1}{4y-1}\,dy + \lim_{t\to\frac{1}{4}^+} \int_t^1 \frac{1}{4y-1}\,dy$ (*)

$= \lim_{t\to\frac{1}{4}^-} \int_{y=0}^{y=t} \frac{1}{4u}\,du + \lim_{t\to\frac{1}{4}^+} \int_{y=t}^{y=1} \frac{1}{4u}\,du$ where $u=4y-1$ $du=4dy$

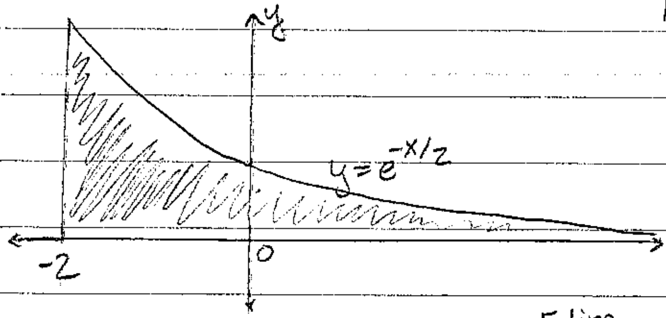
$= \lim_{t\to\frac{1}{4}^-} \left[\frac{1}{4}\ln|4y-1|\right]_0^t + \lim_{t\to\frac{1}{4}^+} \left[\frac{1}{4}\ln|4y-1|\right]_t^1$

$= \lim_{t\to\frac{1}{4}^-} \frac{\ln|4t-1|}{4} - \frac{\ln|1-4(0)|}{4} + \frac{\ln|4(1)-1|}{4} - \lim_{t\to\frac{1}{4}^+} \frac{\ln|4t-1|}{4}$

$= -\infty - \frac{\ln 1}{4} + \frac{\ln 3}{4} - -\infty = -\infty + \infty + \frac{\ln 3}{4}$

$\Rightarrow \int_0^1 \frac{1}{4y-1}dy$ is $\boxed{\text{divergent}}$

③ Sketch the region and find its area, if the area is finite
$S = \{(x,y) \mid x \geq -2,\ 0 \leq y \leq e^{-x/2}\}$



$$\text{Area} = \int_{-2}^{\infty} e^{-x/2}\,dx \qquad u = -x/2 \quad du = -\tfrac{1}{2}dx$$

$$= \int_{x=-2}^{x=\infty} -2e^{u}\,du$$

$$= -2 \lim_{t\to\infty} \left[e^{-x/2}\right]_{-2}^{t}$$

$$= -2\left[\lim_{t\to\infty} e^{-t/2} - e^{+2/2}\right]$$

$$= -2 \lim_{t\to\infty} e^{-t/2} + 2e = \boxed{2e}$$

④ We know from Example 1 that the region $\mathcal{R} = \{(x,y) \mid x \geq 1,\ 0 \leq y \leq 1/x\}$ has infinite area. Show that by rotating $\mathcal{R}$ about the x-axis we obtain a solid with finite volume.

$$\text{Volume} = \int_{1}^{\infty} \pi \left(\frac{1}{x}\right)^2 dx = \pi \lim_{t\to\infty} \int_{1}^{t} \frac{dx}{x^2} = \pi \lim_{t\to\infty} \left[\frac{-1}{x}\right]_{1}^{t}$$

$$= \pi \left[\lim_{t\to\infty} \frac{-1}{t} - \frac{-1}{1}\right] = \pi \lim_{t\to\infty} 1 - \frac{1}{t} = \boxed{\pi < \infty}$$

⑤ Use the Comparison Theorem to determine whether the integral is convergent or divergent. $\int_{1}^{\infty} \frac{2+e^{-x}}{x}\,dx$

Note that for $x \geq 1$ $\quad \frac{2+e^{-x}}{x} > \frac{2}{x} > \frac{1}{x}$ (since $e^{-x} > 0$)

Using equation 2 with $p=1$ $(1 \leq 1)$, we know that $\int_{1}^{\infty} \frac{1}{x}\,dx$ is divergent. So by the Comparison Theorem

$\int_{1}^{\infty} \frac{2+e^{-x}}{x}\,dx$ is $\boxed{\text{divergent.}}$

6. Use the Comparison Theorem to determine whether the integral is convergent or divergent. $\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$

When $x \geq 1$, $0 < \frac{x}{\sqrt{1+x^6}} < \frac{x}{\sqrt{x^6}} = \frac{x}{x^3} = \frac{1}{x^2}$

By equation 2, with $p = 2 > 1$, we know that $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent. So by the Comparison Theorem, $\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$ is $\boxed{\text{convergent.}}$

7. Find a formula for the general term, a_n, of the sequence, assuming that the pattern of the first few terms continues.

$\left\{ -\frac{1}{4}, \frac{2}{9}, \frac{-3}{16}, \frac{4}{25}, \ldots \right\}$

Note that the numerator is n and the denominator is $(n+1)^2$.

Note, also, that the signs of the terms alternate. So,

$$\boxed{a_n = (-1)^n \frac{n}{(n+1)^2}}$$

8. Determine whether the sequence converges or diverges. If it converges, find the limit. $a_n = \frac{n+1}{3n-1}$

Dividing through by n, we have $a_n = \frac{1 + 1/n}{3 - 1/n}$.

Then $\lim_{n\to\infty} \frac{1 + 1/n}{3 - 1/n} = \frac{1+0}{3-0} = \frac{1}{3}$. So $a_n = \frac{n+1}{3n-1}$ $\boxed{\text{converges,}}$ $\boxed{\text{and the limit is } 1/3.}$

(9) Determine whether the sequence converges or diverges. If it converges, find the limit. $a_n = \dfrac{(-1)^n n^3}{n^3+2n^2+1}$

So $|a_n| = \dfrac{n^3}{n^3+2n^2+1}$. Dividing through by n^3, $|a_n| = \dfrac{1}{1+2/n+1/n^3}$

$\lim_{n\to\infty} \dfrac{1}{1+2/n+1/n^3} = \dfrac{1}{1+0+0} = 1$. However, $|a_n| \to 1$ as $n\to\infty$, but

the signs of a_n alternate. So, since $|a_n| = a_n$ when n is even, and $|a_n| = -a_n$ when n is odd, $a_2, a_4, a_6 \ldots$ each $\to 1$ as $n\to\infty$, but $a_1, a_3, a_5, \ldots$ each $\to -1$ as $n\to\infty$. Since the sequence does not approach a single number, it $\boxed{\text{diverges.}}$

(10) Determine whether the sequence converges or diverges. If it converges, find the limit. $\left\{\dfrac{\ln n}{\ln 2n}\right\}$

$a_n = \dfrac{\ln n}{\ln 2n} = \dfrac{\ln n}{\ln 2 + \ln n} = \dfrac{1}{\frac{\ln 2}{\ln n}+1} \to \dfrac{1}{0+1}$ as $n\to\infty$ (since $\ln n \to \infty$ also)

So $a_n \to 1$ as $n\to\infty$, and a_n $\boxed{\text{converges and the limit is 1.}}$

(11) Determine whether the sequence converges or diverges. If it converges, find the limit. $a_n = \dfrac{\sin 2n}{1+\sqrt{n}}$

Since $|\sin 2n| \le 1$, $|a_n| \le \dfrac{1}{1+\sqrt{n}}$. $\lim_{n\to\infty} \dfrac{1}{1+\sqrt{n}} = 0$ $\lim_{n\to\infty} \dfrac{-1}{1+\sqrt{n}} = 0$.

We know $\dfrac{-1}{1+\sqrt{n}} \le a_n \le \dfrac{1}{1+\sqrt{n}}$. So by the Squeeze Theorem,

a_n $\boxed{\text{converges and } \lim_{n\to\infty} a_n = 0}$

12. If \$1000 is invested at 6% interest, compounded annually, then after n years, the investment is worth $a_n = 1000(1.06)^n$ dollars.
(a) Find the first five terms of the sequence $\{a_n\}$.
(b) Is the sequence convergent or divergent? Explain.

(a) $a_n = 1000(1.06)^n$ $a_1 = 1060$ $a_2 = 1123.60$ $a_3 = 1191.02$
$a_4 = 1262.48$ $a_5 = 1338.23$

(b) $\lim_{n\to\infty} a_n = 1000 \lim_{n\to\infty}(1.06)^n$ We know by equation 8 on page 743, that $\lim_{n\to\infty}(1.06)^n$ diverges since $r = 1.06 > 1$. So $\{a_n\}$ $\boxed{\text{diverges.}}$

13. Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded? $a_n = ne^{-n}$
Let $f(x) = xe^{-x}$. Then $f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x)$. For $x > 1$, $f'(x) < 0$, so $f(x)$ is decreasing for $x > 1$. So the sequence $a_n = ne^{-n}$ is $\boxed{\text{decreasing.}}$ Since $\{a_n\}$ is decreasing, it is $\boxed{\text{bounded above by } a_1 = 1/e.}$ Since $e^x > 0$ $x \in (-\infty, \infty)$, $a_n = ne^{-n} > 0$ and $\{a_n\}$ is $\boxed{\text{bounded below by } 0.}$

14. Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded? $a_n = n + \frac{1}{n}$.
Let $f(x) = x + 1/x$. Then $f'(x) = 1 - 1/x^2$. For $x > 1$, $g'(x) > 0$, so $g(x)$ is increasing for $x > 1$. So the sequence $a_n = n + 1/n$ is $\boxed{\text{increasing}}$
$\lim_{n\to\infty} n + \frac{1}{n} = \infty$, so a_n is $\boxed{\text{unbounded.}}$ Note, though, that since $\{a_n\}$ is increasing, it is $\boxed{\text{bounded below by } a_1 = 1 + 1/1 = 2}$