LECTURE OUTLINE
Dot product and cross product

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Math 8

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Rounding Review

The dot product

Scalar and Vector projection

Rounding 101

We will interpret "approximate the following NUMBER to N digits" to mean that we are we know the NUMBER rounded to N *places*. (This means we replace the NUMBER with the a number in the form $\frac{M}{10^N}$ such that $\left|NUMBER - \frac{M}{10^N}\right| \leq \frac{5}{10^{N+1}}$. Notice this leaves two possibilities when our number is in the form $\frac{K}{10^N} + \frac{5}{10^{N+1}}$. What do you usually do and why?).

Yesterday we found $\int_0^1 \sqrt{1+x^4}dx = 1.090918803 \pm 0.002297794121$, or rather that $\int_0^1 \sqrt{1+x^4}dx$ was somewhere in the interval $[1.08862101, 1.09321660]$ (by using the first 4 terms of the series and the remainder estimate for alternating series). Have we approximated $\int_0^1 \sqrt{1+x^4}dx$ to 2 digits? How about to 3 digits?

Our first approximation (using 3 terms) was the observation that $\int_0^1 \sqrt{1+x^4}dx$ was somewhere in the interval $[1.081303419, 1.090918803]$. Had we already approximated $\int_0^1 \sqrt{1+x^4}dx$ to 2 digits?

Dot Product

Given two vectors $\hat{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\hat{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_3 + a_3b_3$$

Ex.: Let $\vec{a} =< -1, 2, 5 >$ and $\vec{b} =< 2, 2, 7 >$, and find $\vec{a} \cdot \vec{b}$.

basic properties

$$\vec{a} \cdot \vec{a} = |a|^2$$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c})$$

$$(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$$

Ex.: Let $\vec{a} =< -1, 2, 5 >$ and $\vec{b} =< 2, 2, 7 >$, and find $(\vec{a} + 3\vec{b}) \cdot (2\vec{a} + \vec{b})$.

The Big Fact

Letting θ be the angle between $\vec{a}$ and $\vec{b}$, we have

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Ex.: Let $\vec{a} =< -1, 2, 5 >$ and $\vec{b} =< 2, 2, 7 >$, and find the angle between $\vec{a}$ and $\vec{b}$.

Orthogonal

Two non-zero vectors $\vec{a}$ and $\vec{b}$ are orthogonal if $\vec{a} \cdot \vec{b} = 0$.

Ex.: Let $\vec{a} =< -1, 2, 5 >$ and $\vec{b} =< 2, 2, 7 >$, and find a non zero vector orthogonal to $\vec{b}$. Should you be able to find a vector orthogonal to both $\vec{a}$ and $\vec{b}$?

Projection

The projection of $\vec{b}$ onto $\vec{a}$ is

$$proj_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}.$$

The component of $\vec{b}$ in the $\vec{a}$ direction is

$$comp_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}.$$

Ex.: Let $\vec{a} =< -1, 2, 5 >$ and $\vec{b} =< 2, 2, 7 >$, and find the component of $\vec{b}$ in the $\vec{a}$ direction and the projection of $\vec{b}$ onto $\vec{a}$.

Examples

Let $\vec{a} =< -1, 2, 5 >$ and $\vec{b} =< 2, 2, 7 >$. Find a length 3 vector such that its component in the $\vec{b}$ is 2. What is your vectors component in the $\vec{a}$ direction? Is it possible to find a length 3 vector such that its component in the $\vec{b}$ is 2 which is perpendicular to $\vec{a}$?

Cross Product

Given vectors $\vec{a}$ and $\vec{b}$ we define $\vec{a} \times \vec{b}$ to be the unique vector satisfying
(1) $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and to $\vec{b}$ (or zero).
(2) It has length equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.
(3) $\vec{a} \times \vec{b}$ is in the direction determined by the right hand rule going from $\vec{a}$ to $\vec{b}$.

Main Theorem

Let $\hat{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\hat{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a} \times \vec{b}$ equals

$$(a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}.$$

Example: Let $\vec{a} =< -1, 2, 5 >$ and $\vec{b} =< 2, 2, 7 >$ and find $\vec{a} \times \vec{b}$. Find a length 3 vector such that its component in the $\vec{b}$ is 2 which is perpendicular to $\vec{a}$.