Math 8
Due 10/22/04

1) Find the Taylor series for f(x) centered at the given value of a.

$F(x) = \sin x, \quad a = \frac{\pi}{2}$

making our chart we get

$f(x) = \sin x \Rightarrow f(a) = f(\frac{\pi}{2}) = 1$

$f'(x) = \cos x \Rightarrow f'(\frac{\pi}{2}) = 0$

$f''(x) = -\sin x \Rightarrow f''(\frac{\pi}{2}) = -1$

$f'''(x) = -\cos x \Rightarrow f'''(\frac{\pi}{2}) = 0$

$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(\frac{\pi}{2}) = 1$

$f^{(5)}(x) = \cos x \Rightarrow f^{(5)}(\frac{\pi}{2}) = 0$

As in the examples we notice a pattern $f^{(n)}(\frac{\pi}{2}) = 0$ if n is odd and $f^{(2n)} = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd} \end{cases}$

Thus the taylor series of $f(x) = \sin(x)$ and $a = \frac{\pi}{2}$ gives

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n = f(\tfrac{\pi}{2}) + \frac{f'(\frac{\pi}{2})}{1!}(x-\tfrac{\pi}{2}) + \frac{f''(\frac{\pi}{2})}{2!}(x-\tfrac{\pi}{2})^2 + \frac{f'''(\frac{\pi}{2})}{3!}(x-\tfrac{\pi}{2})^3 + \cdots$$

$$= \frac{1}{0!} + \frac{0}{1!}(x-\tfrac{\pi}{2}) + \frac{-1}{2!}(x-\tfrac{\pi}{2})^2 + \frac{0}{3!}(x-\tfrac{\pi}{2})^3 + \frac{1}{4!}(x-\tfrac{\pi}{2})^4 + \frac{0}{5!}(x-\tfrac{\pi}{2})^5 + \frac{-1}{6!}(x-\tfrac{\pi}{2})^6 + \cdots$$

$$= \frac{1}{0!} + \frac{(-1)}{2!}(x-\tfrac{\pi}{2})^2 + \frac{1}{4!}(x-\tfrac{\pi}{2})^4 + \frac{(-1)}{6!}(x-\tfrac{\pi}{2})^6 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x-\tfrac{\pi}{2})^{2n}$$

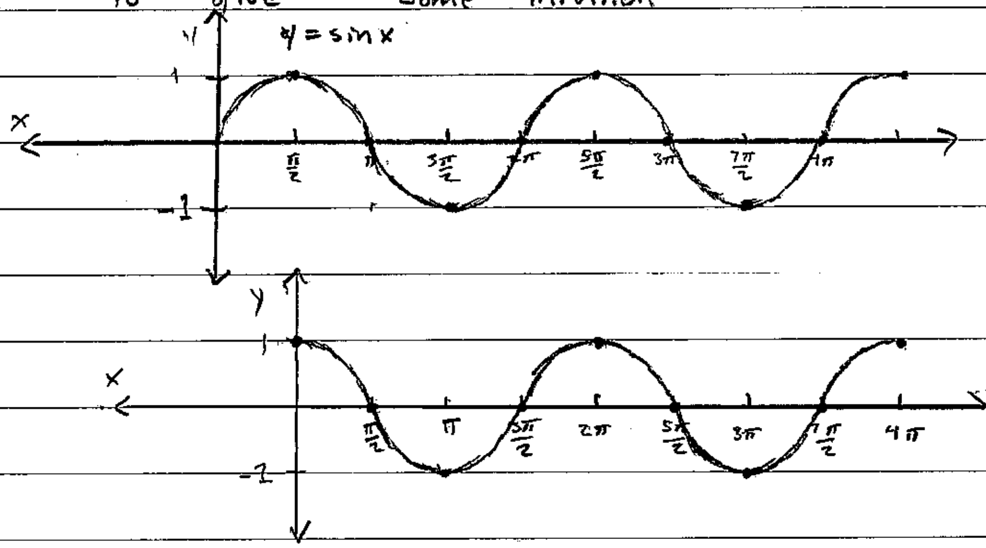
note: This looks remarkably like the Taylor series for cos x at a=0. This similarity will be explained in the alternate solution

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x)^{2n}$$

$a = 0$

Alternate solution for (1)

First we will look at the graphs of sin x and cos x to give some intuition



notice that $\sin(x+\frac{\pi}{2}) = \cos x$ in the graph

we can prove this by noting $\sin(x+y) = \sin x \cos y + \cos x \sin y$

which implies $\sin(x+\frac{\pi}{2}) = \sin x \cos\frac{\pi}{2} + \cos x \sin\frac{\pi}{2} = \cos x$

Now the talyor series of $\sin x$ at $a=\frac{\pi}{2}$ is

the same as $\sin(u+\frac{\pi}{2})$ where $u = x-\frac{\pi}{2}$ at $a=0$

we know $\sin(u+\frac{\pi}{2}) = \cos u$

which implies the taylor series of $\sin(u+\frac{\pi}{2})$ is the same

as $\cos u$ which we know

$$\cos u = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(u)^{2n} \quad \text{but} \quad u = x - \frac{\pi}{2}$$

thus $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x-\frac{\pi}{2})^{2n}$ at $a = \frac{\pi}{2}$

2) Prove that the series obtained in Exercise 1 represents $\sin x$ for all x

First we have the series in (1)
is $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x - \frac{\pi}{2})^{2n}$

As in example 4 in the book
since $(f^{(n+1)}(x))$ is $\pm \sin x$ or $\pm \cos x$ we know that
$|f^{(n+1)}(x)| \leq 1$ for all x
So we can take $m = 1$ in Taylor's inequality:
$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} = |x - \pi/2|^{n+1} / (n+1)!$

by equation 10 we have
$\lim_{n \to \infty}$ of $\frac{|x - \frac{\pi}{2}|^{n+1}}{(n+1)!} = 0$ for all x

Thus $\sin x$ is equal to its Taylor series at $a = \frac{\pi}{2}$
for all x by theorem 8.

3) Use series to approximate the definite integral to within the indicated accuracy.

$$\int_0^{1/2} x^2 e^{-x^2}\,dx \qquad (|\text{error}| < 0.001)$$

as in example 8

we know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ which implies

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{n!}$$

hence $x^2 e^{-x^2} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+2}}{n!}$

Now we integrate term by term

$$\int \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+2}}{n!}\,dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n (x)^{2n+2}}{n!}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+3}}{(2n+3)\,n!}$$

This series converges for all x since the series for $x^2 e^{-x^2}$ converges for all x

The evaluation theorem gives

$$\int_0^{1/2} x^2 e^{-x^2}\,dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+3}}{(2n+3)\,n!} \right]_0^{1/2} = \frac{(\frac{1}{2})^3}{3} - \frac{(\frac{1}{2})^5}{5} + \frac{(\frac{1}{2})^7}{(7)2!} - \cdots$$

The alternating series estimation theorem gives

$|R_n| = |s - s_n| \leq b_{n+1}$

so we need

$$\frac{(\frac{1}{2})^n}{(2n+3)\,n!} < \frac{1}{1000} \qquad \text{certainly if } n = 4$$

from example 8

then $\frac{1}{16 \cdot 11 \cdot 4!} < \frac{1}{11 \cdot 5!} = \frac{1}{1320} < 0.001$

Thus by the alternating series estimation theorem

$$\sum_{n=0}^{3} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+3}}{(2n+3)n!}$$ gives an estimate of

$$\int_0^{1/2} x^2 e^{-x^2}$$ with $|\text{error}| < 0.001$

$$\sum_{n=0}^{3} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+3}}{(2n+3)n!} = \frac{\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{2}\right)^5}{5} + \frac{\left(\frac{1}{2}\right)^7}{7 \cdot 2!} - \frac{\left(\frac{1}{2}\right)^9}{9 \cdot 3!}$$

$$\approx .035939$$

4) Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n\, n!} = \sum_{n=0}^{\infty} \frac{\left(\frac{3}{5}\right)^n}{n!} = e^{3/5}$$

since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

5) (a) Approximate f by a Taylor Polynomial with degree n at the number a

(b) Use Taylor's inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval

$f(x) = \cos x, \quad a = \pi/3 \quad n = 4, \quad 0 \le x \le \frac{\pi}{6}$

(a)

$$\begin{aligned}
f(x) &= \cos x &\Rightarrow\quad f\left(\tfrac{\pi}{3}\right) &= \tfrac{1}{2} \\
f'(x) &= -\sin x &\Rightarrow\quad f'\left(\tfrac{\pi}{3}\right) &= -\tfrac{\sqrt{3}}{2} \\
f''(x) &= -\cos x &\Rightarrow\quad f''\left(\tfrac{\pi}{3}\right) &= -\tfrac{1}{2} \\
f'''(x) &= \sin x &\Rightarrow\quad f'''\left(\tfrac{\pi}{3}\right) &= \tfrac{\sqrt{3}}{2} \\
f^{(4)}(x) &= \cos x &\Rightarrow\quad f^{(4)}\left(\tfrac{\pi}{3}\right) &= \tfrac{1}{2}
\end{aligned}$$

So $T_4 = \sum_{n=0}^{4} \frac{f^{(n)}(\frac{\pi}{3})}{n!}\left(x - \frac{\pi}{3}\right)^n = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{1}{1!}\right)\left(x - \frac{\pi}{3}\right) - \frac{1}{2}\left(\frac{1}{2!}\right)\left(x - \frac{\pi}{3}\right)^2$

$+ \frac{\sqrt{3}}{2}\left(\frac{1}{3!}\right)\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{2}\left(\frac{1}{4!}\right)\left(x - \frac{\pi}{3}\right)^4$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{3}\right)^4$

Note: Notice this f is the derivative of the function in example 12.10.7, so you can differentiate that example to arrive at the answer.

(b) As in example 2

Since $|\sin x|, |\cos x| \le 1$ we can take, $M = 1$

Thus $|R_4(x)| \le \frac{1}{5!}\left|x - \frac{\pi}{3}\right|^5 \le \frac{1}{5!}\left|0 - \frac{\pi}{3}\right|^5$ on $\left[0, \frac{\pi}{6}\right]$

$\approx .010495$

6) Use the information from problem 5 to estimate $\cos 69^\circ$ correct to five decimal places.

From problem 5 we have

$$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right) - \frac{1}{4}\left(x-\frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x-\frac{\pi}{3}\right)^3 + \frac{1}{48}\left(x-\frac{\pi}{3}\right)^4 + R_4(x)$$

Now $x = 69^\circ = (60^\circ + 9^\circ) = \left(\frac{\pi}{3} + \frac{\pi}{20}\right)$ radians

Thus by the taylor estimation

$$|R_4(x)| \le \frac{1}{5!}\left(x-\frac{\pi}{3}\right)^5 = \frac{1}{5!}\left(\frac{\pi}{20}\right)^5 \approx 7.96926 \times 10^{-7}$$

$$< 8 \times 10^{-7} < 10^{-5}$$

Therefore our estimate using T_4 will give an answer accurate to five decimal places

$$\cos\left(\frac{\pi}{3} + \frac{\pi}{20}\right) \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{20}\right) - \frac{1}{4}\left(\frac{\pi}{20}\right)^2 + \frac{\sqrt{3}}{12}\left(\frac{\pi}{20}\right)^3 + \frac{1}{48}\left(\frac{\pi}{20}\right)^4$$

$$\approx 0.35837$$

7) How many terms of the Maclaurian series for $\ln(1+x)$ do you need to estimate $\ln(1.4)$ to within 0.001

We have calculated in a previous homework that the Maclaurian series for $\ln(1+x)$ is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$.

since this is an alternating series we can use the alternating series estimation theorem

ie $|R_n| \leq b_{n+1}$

$\ln(1.4) = \ln(1+0.4) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(\frac{2}{5})^n}{n}$

we need to find n s.t

$\frac{(\frac{2}{5})^n}{n} < \frac{1}{1000}$ $\qquad \frac{(\frac{2}{5})^n}{n} \leq \left(\frac{2}{5}\right)^n$

so $\left(\frac{2}{5}\right)^n < \frac{1}{1000}$

when $n\ln\left(\frac{2}{5}\right) < \ln\left(\frac{1}{1000}\right)$ $\qquad$ since $\ln\left(\frac{2}{5}\right) < 0$

we have $n > \frac{\ln\left(\frac{1}{1000}\right)}{\ln\left(\frac{2}{5}\right)} = 7.538$:

since $\frac{(\frac{2}{5})^n}{n} \leq \left(\frac{2}{5}\right)^n$ and for $n \geq 8$ $\left(\frac{2}{5}\right)^n < \frac{1}{1000}$

certainly the first 8 will give us the desired accuracy, however we can do better than that

some trial and error with numbers less than 8

gives $|a_6| = (\frac{2}{5})^6/6 \approx 0.0007 < 0.001$

but $|a_5| = (\frac{2}{5})^5/6 \approx 0.002048 > 0.001$

thus we need the first 5 terms of the series to estimate $\ln(1.4)$ to within 0.001.