Homework due 10/20

1. $\arctan(x/3) = \int \frac{1}{3(1+(\frac{x}{3})^2)}dx$

$$= \frac{1}{3}\int \frac{1}{1+(\frac{x}{3})^2}dx$$

$$= \frac{1}{3}\int \sum_{n=0}^{\infty} (-(\tfrac{x}{3})^2)^n dx$$ by comparison with the power series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, $|x|<1$

$$= \frac{1}{3}\int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{3^{2n}} dx$$

$$= \frac{1}{3}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot 3^{2n}} + C$$

Now $0 = \arctan(0/3) = C$

so $\arctan(\frac{x}{3}) = \frac{1}{3}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot 3^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot 3^{2n+1}}$

This converges for $|\frac{x}{3}|<1$, i.e. $|x|<3$.

2. $\int \tan^{-1}(x^2)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)(4n+3)} + C$

$R=1$ by Example 7.

3. (a.) If the series given were correct, we would have (by comparing the general form) that $f'(1) = -0.8$. But the graph of f at $x=1$ is sloping upwards, so we should have $f'(x)>0$.

(b.) At $x=2$, f is near its maximum value. Thus although the slope $f'(2)$ may still be positive, the slope is decreasing, so comparing the second terms we should have $f''(2)<0$.

But the series suggested would give us $\frac{f''(2)}{2} = 1.5$

so $f''(x) = 3$. So this cannot be correct.

4. We have

$$\begin{aligned} f(x) &= \sin(2x) & f(0) &= 0 \\ f'(x) &= 2\cos(2x) & f'(0) &= 2 \\ f''(x) &= -4\sin(2x) & f''(0) &= 0 \\ f^{(3)}(x) &= -8\cos(2x) & f^{(3)}(0) &= -8 \end{aligned}$$

etc.

so
$$\sin(2x) = \frac{2}{1!}x - \frac{2^3}{3!}x^3 + \frac{2^5 x^5}{5!} - \dots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}$$

Now $|a_n| = \left|\frac{(2x)^{2n+1}}{(2n+1)!}\right|$ so $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2x)^{2n+3}\cdot(2n+1)!}{(2n+3)!\cdot(2x)^{2n+1}}\right| = \left|\frac{(2x)^2}{(2n+3)(2n+2)}\right|$

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 \quad \text{for all } x.$$

so the Maclaurin series is valid for all x.

5.

$$\begin{aligned} f(x) &= x^3 & f(-1) &= -1 \\ f'(x) &= 3x^2 & f'(-1) &= 3 \\ f''(x) &= 6x & f''(-1) &= -6 \\ f^{(3)}(x) &= 6 & f^{(3)}(-1) &= 6 \end{aligned}$$

and clearly $f^{(n)}(x) = 0$ for $n > 3$.

So
$$x^3 = -1 + 3(x+1) - \frac{6}{2!}(x+1)^2 + \frac{6}{3!}(x+1)^3$$
$$= -1 + 3(x+1) - 3(x+1)^2 + (x+1)^3.$$

6.

$$f(x) = \ln x \qquad f(2) = \ln 2$$

$$f'(x) = \frac{1}{x} \qquad f'(2) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{x^2} \qquad f''(2) = -\frac{1}{4}$$

$$f^{(3)}(x) = \frac{2}{x^3} \qquad f^{(3)}(2) = \frac{1}{4}$$

$$f^{(4)}(x) = \frac{-6}{x^4} \qquad f^{(4)}(2) = \frac{-3}{8}$$

In fact, $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$ $\qquad f^{(n)}(2) = \frac{(-1)^{n-1}(n-1)!}{2^n}$

so $$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{2^n} \cdot \frac{1}{n!}(x-2)^n$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 2^n}(x-2)^n$$

7. We have $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

so $$\sin(x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+4}}{(2n+1)!}$$

8. We have $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots$

so $$e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

so $$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$$

$$\therefore \int \frac{e^x - 1}{x}\,dx = \int \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}\,dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!\cdot(n+1)} + C$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n!\cdot n} + C$$

9.

$$\sec x = \frac{1}{\cos x} = \frac{1}{1-\frac{x^2}{2}+\frac{x^4}{4!}-\dots}$$

$$\begin{array}{r l}
 & 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \dots \\
\hline
1-\frac{x^2}{2}+\frac{x^4}{24}-\dots \,\Big) & 1 \\
 & 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots \\
\hline
 & \frac{x^2}{2} - \frac{x^4}{24} + \dots \\
 & \frac{x^2}{2} - \frac{x^2}{2}\cdot\frac{x^2}{2} + \dots \\
\hline
 & \left(-\frac{x^4}{24}+\frac{x^4}{4}\right) + \dots \\
 & \parallel \\
 & \frac{5x^4}{24}
\end{array}$$

10.

$$e^x \cdot \ln(1-x) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\cdot\left(\sum_{n=1}^{\infty} \frac{-x^n}{n}\right) \quad \text{by §12.9 Example 6.}$$

$$= \left(1+x+\frac{x^2}{2}+\dots\right)\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\dots\right)$$

$$= -x + \left(-x^2-\frac{x^2}{2}\right) + \left(-\frac{x^3}{2}-\frac{x^3}{2}-\frac{x^3}{3}\right) + \dots$$

$$= -x - \frac{3}{2}x^2 - \frac{4}{3}x^3 + \dots$$