1) Evaluate the integral

$$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx$$

The method of partial fractions gives:

$$\frac{x-1}{x^2+3x+2} = \frac{x-1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$

thus $A(x+1) + B(x+2) = x-1$

$(A+B)x + A + 2B = x - 1$

thus $A + B = 1$ and $A + 2B = -1$

$A = 1 - B$ giving $-1 = A + 2B = 1 - B + 2B = 1 + B$

thus $B = -2$

and $A = 3$

giving $$\frac{x-1}{x^2+3x+2} = \frac{3}{x+2} + \frac{-2}{x+1}$$

which implies

$$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = \int_0^1 \left(\frac{3}{x+2} + \frac{-2}{x+1}\right) dx = 3\int_0^1 \frac{dx}{x+2} - 2\int_0^1 \frac{dx}{x+1}$$

$$= \Big[3\ln|x+2| - 2\ln|x+1|\Big]\Big|_0^1$$

$$= 3\ln|1+2| - 2\ln|1+1| - (3\ln|0+2| - 2\ln|0+1|)$$

$$= 3\ln(3) - 2\ln(2) - 3\ln(2) + 2\underbrace{\cancel{\ln(1)}}_{0}$$

$$= 3\ln(3) - 5\ln(2) = \ln\left(\tfrac{27}{32}\right)$$

2) Evaluate the integral

$$\int \frac{x^2 - x + 6}{x^3 + 3x}\, dx$$

The method of partial fractions gives:

$$\frac{x^2 - x + 6}{x^3 + 3x} = \frac{x^2 - x + 6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx + C}{x^2+3}$$

thus $A(x^2+3) + (Bx+C)x = x^2 - x + 6$

$(A+B)x^2 + Cx + 3A = x^2 - x + 6$

giving $3A = 6$ $\quad C = -1 \quad$ $A + B = 1$

implying $A = 2$ $\quad$ $1 = A + B = 2 + B$

implying $B = -1$

giving $$\frac{x^2 - x + 6}{x^3 + 3x} = \frac{2}{x} + \frac{-x + (-1)}{x^3 + 3} = \frac{2}{x} - \frac{x+1}{x^2+3}$$

which implies

$$\int \frac{x^2 - x + 6}{x^3 + 3x}\, dx = \int \left(\frac{2}{x} - \frac{x+1}{x^2+3} \right) dx = \int \left(\frac{2}{x} - \frac{x}{x^2+3} - \frac{1}{x^2+3} \right) dx$$

$$= 2 \int \frac{dx}{x} - \int \frac{x}{x^2+3}\, dx - \int \frac{dx}{x^2+3}$$

$$= 2 \ln|x| - \int \frac{x}{x^2+3}\, dx - \frac{\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)}{\sqrt{3}}$$

let $u = x^2 + 3 \qquad du = 2x\, dx$

$$= 2 \ln|x| - \frac{1}{2} \int \frac{du}{u} - \frac{\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)}{\sqrt{3}}$$

$$= 2 \ln|x| - \frac{1}{2} \ln|u| - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

$$= 2 \ln|x| - \frac{1}{2} \ln|x^2+3| - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

$$= \ln\left| x^2 / \sqrt{x^2+3} \right| - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

3) Evaluate the integral

$$\int_0^1 \frac{x}{x^2+4x+13}\,dx$$

complete the square $x^2+4x+13 = x^2+4x+4+9 = (x+2)^2+9$

giving $\dfrac{x}{x^2+4x+13} = \dfrac{x}{(x+2)^2+9}$

thus $\displaystyle\int_0^1 \frac{x}{x^2+4x+13}\,dx = \int_0^1 \frac{x\,dx}{(x+2)^2+9}$ let $u = x+2$ $du = dx$

$x = u-2$

$$= \int_{u=(0)+2=2}^{u=1+2=3} \frac{u-2}{u^2+9}\,du = \int_2^3 \frac{u}{u^2+9}\,du - 2\int_2^3 \frac{du}{u^2+9}$$

Let $t = u^2+9$ $dt = 2u\,du$

$$= \frac{1}{2}\int_{t=2^2+9=13}^{t=3^2+9=18} \frac{dt}{t} - \left[\frac{2}{3}\tan^{-1}\left(\frac{u}{3}\right)\right]\Bigg|_2^3$$

$$= \left[\frac{1}{2}\ln|t|\right]\Bigg|_{13}^{18} - \left[\frac{2}{3}\tan^{-1}(1) - \frac{2}{3}\tan^{-1}\left(\frac{2}{3}\right)\right]$$

$$= \tfrac{1}{2}\ln(18) - \tfrac{1}{2}\ln(13) - \tfrac{2}{3}\tan^{-1}(1) + \tfrac{2}{3}\tan^{-1}\left(\tfrac{2}{3}\right)$$

$$= \ln\left(\sqrt{(18/13)}\right) - \frac{\pi}{6} + \frac{2}{3}\tan^{-1}\left(\frac{2}{3}\right)$$

4) Determine whether the given integral is convergent or divergent. Evaluate if convergent.

$$\int_0^{\infty} \frac{x}{(x^2+2)^2}\, dx$$

There are no points of discontinuity of $\frac{x}{(x^2+2)^2}$ on $[0,\infty)$ since $x^2 \geq 0$ on $[0,\infty]$ which implies $x^2+2 \neq 0$ on $[0,\infty]$

thus $\displaystyle\int_0^{\infty} \frac{x}{(x^2+2)^2}\, dx = \lim_{t\to\infty} \int_0^{t} \frac{x}{(x^2+2)^2}\, dx$ let $u = x^2+2$, $du = 2x\, dx$

$$= \lim_{t\to\infty} \frac{1}{2} \int_2^{t^2+2} \frac{1}{u^2}\, du = \lim_{t\to\infty} \frac{1}{2} \left[-\frac{1}{u} \right]\Big|_2^{t^2+2}$$

$$= \lim_{t\to\infty} \frac{1}{2} \left[-\frac{1}{t^2+2} - \left(-\frac{1}{2} \right) \right]$$

$$= \lim_{t\to\infty} \left[\frac{1}{4} - \frac{1}{2(t^2+2)} \right] = \frac{1}{4} - 0 = \frac{1}{4}$$

The improper integral is convergent.

5) Determine whether the integral is convergent or divergent. Evaluate if convergent.

$$\int_{-\infty}^{\infty} x^2 e^{-x^3}\,\partial x$$

There are no points of discontinuity of $x^2 e^{-x^3}$

Next $\displaystyle\int_{-\infty}^{\infty} x^2 e^{-x^3}\partial x = \lim_{t\to-\infty}\int_t^0 x^2 e^{-x^3}\partial x + \lim_{t\to\infty}\int_0^t x^2 e^{-x^3}\partial x$

Now $\displaystyle\lim_{t\to-\infty}\int_t^0 x^2 e^{-x^3}\partial x$ let $u = x^3$ $\partial u = 3x^2\,\partial x$

giving $\displaystyle\lim_{t\to-\infty}\int_t^0 x^2 e^{-x^3}\partial x = \lim_{t\to-\infty}\frac{1}{3}\int_{t^3}^{0} e^{-u}\,\partial u$

$$= \lim_{t\to-\infty}\frac{1}{3}\left(-e^{-u}\right)\Big|_{t^3}^{0} = \lim_{t\to-\infty}\frac{1}{3}\left[e^{-t^3} - 1\right]$$

$= \infty$

$\Rightarrow \displaystyle\int_{-\infty}^{\infty} x^2 e^{-x^3}\partial x$ is divergent

6) Find the values of p for which the integral converges and evaluate the integral for those values of p.

$$\int_c^\infty \frac{dx}{x(\ln x)^p} \qquad c>0$$

Let $u = \ln x \qquad du = \frac{1}{x}\,dx$

thus $$\int_c^\infty \frac{dx}{x(\ln(x))^p} = \lim_{t\to\infty} \int_{\ln(c)}^{\ln(t)} \frac{du}{u^p}$$

$$= \lim_{t\to\infty} \begin{cases} -\left(\frac{1}{p-1} \cdot \frac{1}{u^{p-1}}\right) & p \neq 1 \\ \ln(u) & p=1 \end{cases} \Bigg|_{\ln(c)}^{\ln(t)} \qquad \text{Now } \ln(t)\to\infty \text{ as } t\to\infty$$

$$= \lim_{t\to\infty} \begin{cases} \frac{1}{1-p}\frac{1}{(\ln(t))^{p-1}} - \frac{1}{1-p}\left(\frac{1}{(\ln(c))^p}\right) & p\neq 1 \\ \ln(\ln(t)) - \ln(\ln(c)) & p=1 \end{cases}$$

Now if $p>1$ then $p-1>0$ (As in example 4)

so as $t\to\infty$, $\ln(t)\to\infty$, and $(\ln(t))^{p-1}\to\infty$

implying $\frac{1}{(\ln(t))^{p-1}} \to 0$ for $p>1$

Therefore if $p>1$, $\int^\infty \frac{dx}{x(\ln(x))^p} = \lim_{t\to\infty}\left[\frac{1}{1-p}\frac{1}{(\ln(t))^{p-1}} - \frac{1}{[1-p]}\frac{1}{(\ln(c))^p}\right]$

$$= \left(\frac{1}{p-1}\right)\frac{1}{(\ln(c))^p}$$

if $p<1$ then $p-1<0$

so as $t\to\infty$ $\ln(t)\to\infty$ $(\ln(t))^{p-1}\to 0$

implying $\frac{1}{(\ln(t))^{p-1}}\to\infty$ Diverges for $p<1$

if $p=1$ then as $t\to\infty$ $\ln(t)\to\infty$ implying $\ln(\ln(t))\to\infty$

thus $\int_c^\infty \frac{dx}{x(\ln x)^p}$ diverges for $p=1$

Hence $\displaystyle\int_1^{\infty} \frac{dx}{x(\ln(x))^p}$ converges for $p \in (1, \infty)$ and diverges otherwise

for $p \in (1, \infty)$

$$\int_1^{\infty} \frac{dx}{x(\ln(x))^p} = \frac{1}{(p-1)(\ln(c))^p}$$