Homework #8 Due: October 11, 2004

① Test the series for convergence or divergence.

$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n b_n$ where $b_n = \frac{\sqrt{n}}{1+2\sqrt{n}}$

Find $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{1+2\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+2} = \frac{1}{2} \neq 0$.

So $\lim_{n\to\infty} (-1)^n \frac{\sqrt{n}}{1+2\sqrt{n}} \neq 0$, the series $\boxed{\text{diverges.}}$ (Note
by the Test for Divergence

that, in fact, $\lim_{n\to\infty} (-1)^n \frac{\sqrt{n}}{1+2\sqrt{n}}$ does not exist.)

② Test the series for convergence or divergence.

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$ We can rewrite this limit as

$0 + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$. Let $b_n = \frac{\ln n}{n} > 0$ for $n \geq 2$.

Let $f(x) = \frac{\ln x}{x}$. Then $f'(x) = \frac{1-\ln x}{x^2} < 0$ when

$x > e$. So we know that $\{b_n\}$ is eventually decreasing. Also, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x}$

which, by L'Hopital's Rule is equal to $\lim_{x\to\infty} \frac{1/x}{1} = 0$.

So, by the Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$ $\boxed{\text{converges.}}$

3 Test the series for convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!}$$

We know that when n is even, $\sin(n\pi/2) = 0$.
Also when $n = 2k+1$, $\sin(n\pi/2) = (-1)^k$.
So $\sum_{n=1}^{\infty} \frac{\sin \pi n/2}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$. Here $b_n = \frac{1}{(2n+1)!} > 0$.
Also, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$, so
the series $\boxed{\text{converges}}$ by the Alternating Series Test.

4 How many terms of the series do we need to add in order to find the sum to the indicated accuracy?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \quad (|\text{error}| < 0.001)$$

We know the series converges by the Alternating series test because $b_{n+1} = \frac{1}{(n+1)^4} < \frac{1}{n^4} = b_n$ and because

$\lim_{n\to\infty} 1/n^4 = 0$. Now we use the Alternating Series Estimation Theorem. $b_5 = \frac{1}{5^4} = 0.0016 > 0.001$

and $b_6 = \frac{(-1)^{6+1}}{6^4} \approx 0.00077 < 0.001$

So we need $\boxed{5}$ terms.

5 For what values of p is each series convergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$$

Consider $p>0$, $\frac{1}{(n+1)^p} \leq \frac{1}{n^p}$ (So, $\frac{1}{n^p}$ is decreasing.)

and $\lim_{n\to\infty} \frac{1}{n^p} = 0$, so the series converges by the Alternating Series Test. Now consider $p \leq 0$, $\lim_{n\to\infty} \frac{(-1)^{n-1}}{n^p}$ does not exist, so the series diverges by the Test for Divergence. So $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges only when $\boxed{p>0}$.

6 Determine whether the series is absolutely convergent, conditionally convergent, or divergent. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a convergent p-series with $p=4>1$, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ is $\boxed{\text{absolutely convergent.}}$

7 Determine whether the series is absolutely convergent, conditionally convergent, or divergent. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{5+n}$ $\quad a_n = (-1)^n \frac{n}{5+n}$

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{5+n} = \lim_{n\to\infty} \frac{1}{\frac{5}{n}+1} = 1$. So $\lim_{n\to\infty} a_n \neq 0$.

So $\sum_{n=1}^{\infty} (-1)^n \frac{n}{5+n}$ is $\boxed{\text{divergent}}$ by the Test for Divergence

8. Determine whether the series is absolutely convergent, conditionally convergent, or divergent. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$

We will apply the Limit Comparison Test with the harmonic series to this series: $\lim_{n\to\infty} \frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$.

So $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is divergent with this test.
But, with the Alternating Series test, we have that $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ converges since ① $\left\{\frac{n}{n^2+1}\right\}$ has positive terms
② is decreasing since $\left(\frac{x}{x^2+1}\right)' = \frac{1-x^2}{(x^2+1)^2} \le 0$ for $x \ge 1$

and ③ $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$. So $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ is convergent, but not absolutely convergent. We call this $\boxed{\text{conditionally convergent}}$

9. Determine whether the series is absolutely convergent, conditionally convergent, or divergent. $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$

We know that $\left|\frac{\sin 4n}{4^n}\right| \le \frac{1}{4^n}$, so $\sum_{n=1}^{\infty} \left|\frac{\sin 4n}{4^n}\right|$ converges by comparison with the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{4^n}$ $\left(|r| = \frac{1}{4} < 1\right)$

So $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ is $\boxed{\text{absolutely convergent.}}$

10 Determine whether the series is absolutely convergent, conditionally convergent, or divergent. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln n}$

We know that $\left\{\frac{1}{n \ln n}\right\}$ is decreasing and $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$.

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln n}$ converges by the Alternating Series Test.

$\int_1^{\infty} \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} (\ln(\ln t) - \ln(\ln(1)))] = \infty$ So $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$

diverges by the Integral test. So, since $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln n}$

converges, but not absolutely, it is $\boxed{\text{conditionally convergent.}}$