Homework due November 8

1. Find an equation of the plane through the origin and the points $(2,-4,6)$ & $(5,1,3)$

Since $(0,0,0)$, $(2,-4,6)$, & $(5,1,3)$ lie in the plane, we know that $\langle 2,-4,6\rangle$ & $\langle 5,1,3\rangle$ lie in the plane. So to find the normal vector to the plane, we take the cross product. $\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 6 \\ 5 & 1 & 3 \end{vmatrix} = (-12-6)\vec{i} + (30-6)\vec{j} + (2--20)\vec{k}$
$= -18\vec{i} + 24\vec{j} + 22\vec{k}$

So the equation of the plane is
$-18(x-0) + 24(y-0) + 22(z-0) = 0$ or $-18x + 24y + 22z$.

2. (a) Find symmetric equations for the line of intersection of the planes. plane a: $x-2y+z=1$ plane b: $2x+y+z=1$
$\Rightarrow$ $\vec{n}_a = \langle 1,-2,1\rangle$ $\vec{n}_b = \langle 2,1,1\rangle$
The line of intersection is $\perp$ to both $\vec{n}_a$ & $\vec{n}_b$. So to find the direction of the line of intersection we find
$\vec{n}_a \times \vec{n}_b = \begin{vmatrix} \vec{i} & \vec{j} & k \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = (-2-1)\vec{i} + (2-1)\vec{j} + (1--4)\vec{k}$
$= \langle -3, 1, 5\rangle$

We also need to find a point on the line. Let $x=y=0$. Then $z=1$ & $(0,0,1)$ are on both planes, hence on the line of intersection. So $\vec{l} = \langle 0,0,1\rangle + t\langle -3,1,5\rangle$ and $\frac{x}{-3} = y = \frac{z-1}{5}$

(b) Find the angle between the planes.
To do this, we find the angle between the normal vectors.
$\cos\theta = \frac{\vec{n}_a \cdot \vec{n}_b}{|\vec{n}_a||\vec{n}_b|} = \frac{2-2+1}{\sqrt{6}\sqrt{6}} = \frac{1}{6} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{6}\right) \approx 80°$

3. Find the distance from $(3,-2,7)$ to $4x-6y+z=5$
We use eq'n 9. $D = \frac{1}{\sqrt{16+36+1}}\left(4(3) + (-6)(-2) + (7)(1) - 5\right) = \frac{26}{\sqrt{53}}$

3. Find the limit. $\lim_{t\to 0}\left\langle \frac{e^t-1}{t}, \frac{\sqrt{1+t}-1}{t}, \frac{3}{1+t}\right\rangle$

Consider each limit separately. $\lim_{t\to 0}\frac{e^t-1}{t} \overset{H}{=} \lim_{t\to 0}\frac{e^t}{1} = 1$

$\lim_{t\to 0}\frac{\sqrt{1+t}-1}{t} = \lim_{t\to 0}\frac{\sqrt{1+t}-1}{t}\cdot\frac{\sqrt{1+t}+1}{\sqrt{1+t}+1} = \lim_{t\to 0}\frac{1}{\sqrt{1+t}+1} = \frac{1}{2}$

$\lim_{t\to 0}\frac{3}{1+t} = 3$ So $\lim_{t\to 0}\left\langle \frac{e^t-1}{t}, \frac{\sqrt{1+t}-1}{t}, \frac{3}{1+t}\right\rangle = \langle 1, 1/2, 3\rangle$

4. Find a vector equation and parametric equations for the line segment that joins P to Q.

$P(1,0,1)$, $Q(2,3,1)$ Let $\vec{r}_0 = \langle 1,0,1\rangle$ & $\vec{r}_1 = \langle 2,3,1\rangle$ and use formula 13.5.4.

$\vec{r}(t) = (1-t)\langle 1,0,1\rangle + t\langle 2,3,1\rangle = \langle 1-t+2t, 3t, 1-t+t\rangle$

$\Rightarrow \vec{r}(t) = \langle 1+t, 3t, 1\rangle$ or $x = 1+t$, $y = 3t$, $z = 1$

5-10. Match the parametric equations with the graphs. Give reasons.

5. VI We note that $x^2+z^2 = \cos^2 4t + \sin^2 4t = 1$ It's a circular cylinder about the y-axis. So the curve lies on this cylinder. Since $y=t$, we have a helix.

6. II Since $x=t$ & $y=t^2$, $y=x^2$. So the curve lies on the parabolic cylinder $y=x^2$. Also note that y & z are always positive. Consider $\lim_{t\to\infty}$ & $\lim_{t\to-\infty}$. $\lim_{t\to\infty}\langle t, t^2, e^{-t}\rangle = \langle\infty,\infty,0\rangle$ & $\lim_{t\to-\infty}\langle t, t^2, e^{-t}\rangle = \langle -\infty,\infty,\infty\rangle$. So the graph is II.

7. IV Again, note that y & z are always positive. Consider the limits. $\lim_{t\to\infty}\langle t, \frac{1}{1+t^2}, t^2\rangle = \langle\infty, 0, \infty\rangle$ $\lim_{t\to-\infty}\langle t, \frac{1}{1+t^2}, t^2\rangle = \langle -\infty, 0, \infty\rangle$

So the graph is IV

8. I Note that z is always positive. Also note $x^2+y^2 = e^{-2t}\cos^2 10t + e^{-2t}\sin^2 10t = e^{-2t}(\cos^2 10t + \sin^2 10t) = e^{-2t} = (e^{-t})^2 = z^2$. So the curve lies on the cone $x^2+y^2=z^2$. So the graph is I.

(9) V Note that $x^2+y^2=\cos^2 t+\sin^2 t=1$, so the curve lies on a circular cylinder about the z axis. x, y, & z are all periodic and the curve repeats itself. Hence, the graph is V

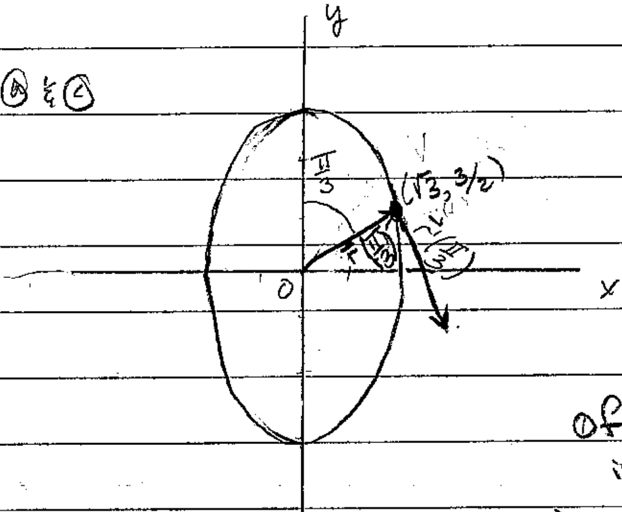
(10) III Note that $x^2+y^2=\cos^2 t+\sin^2 t=1$, so the curve lies on a circular cylinder about the z-axis. Consider the limit: $\lim_{t\to 0} \ln t=-\infty$, so $z\to-\infty$ and the graph is III

(11) $\vec{r}(t)=2\sin t\,\vec{i}+3\cos t\,\vec{j}$ $\quad t=\pi/3$

(a) Sketch the plane curve with the given equation. (b) Find $\vec{r}'(t)$
(c) Sketch the position vector $r(t)$ & the tangent vector $\vec{r}'(t)$ for the given value of t.

$\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=\sin^2 t+\cos^2 t=1$ So we have an ellipse

(a) & (c)



(b) $\vec{r}'(t)=2\cos t\,\vec{i}+-3\sin t\,\vec{j}$

(c) $\vec{r}(\pi/3)=\langle\sqrt{3}, 3/2\rangle$
$r'(\pi/3)=\langle 1, -3\sqrt{3}/2\rangle$
So $\vec{r}(\pi/3)$ is in the direction of $\langle 1, -3\sqrt{3}/2\rangle$, but its endpoint is at $(\sqrt{3}, 3/2)$

(12) Find the unit tangent vector $\vec{T}(t)$ at the point with the given value of the parameter t. $\vec{r}(t)=2\sin t\,\vec{i}+2\cos t\,\vec{j}+\tan t\,\vec{k}$ $\quad t=\pi/4$

$\vec{r}'(t)=2\cos t\,\vec{i}-2\sin t\,\vec{j}+\sec^2 t\,\vec{k}$

$\vec{r}'(\pi/4)=\langle\sqrt{2}, -\sqrt{2}, 2\rangle$ $\quad |\vec{r}'(\pi/4)|=\sqrt{2+2+4}=\sqrt{8}=2\sqrt{2}$

So $\vec{T}(\pi/4)=\frac{1}{2\sqrt{2}}\langle\sqrt{2},-\sqrt{2},2\rangle=\langle\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\rangle$

13 Find the parametric equations for the tangent line to the curve with the given parametric equations at the specified point

$x = t^2 - 1,\ y = t^2 + 1,\ z = t + 1$ @ $(-1, 1, 1)$

$\vec{r}(t) = \langle t^2 - 1, t^2 + 1, t + 1 \rangle$ so $\vec{r}\,'(t) = \langle 2t, 2t, 1 \rangle$

What is the value of t @ $(-1, 1, 1)$

$-1 = t^2 - 1 \quad t^2 = 0 \Rightarrow t = 0$

So the tangent vector @ $(-1, 1, 1)$ is $\vec{r}\,'(0) = \langle 0, 0, 1 \rangle$

So the tangent line is parallel to the tangent vector

$\vec{v}(t) = \langle -1, 1, 1 \rangle + t\langle 0, 0, 1 \rangle$ so $x = -1 \quad y = 1 \quad z = 1 + t$

14 Determine whether the curve is smooth $\vec{r}(t) = \langle t^3, t^4, t^5 \rangle$

$\vec{r}\,'(t) = \langle 3t^2, 4t^3, 5t^4 \rangle$. At $t = 0$, $\vec{r}\,'(t) = \langle 0, 0, 0 \rangle$, so the curve is not smooth.

15 & 16 Find the length of the curve.

15 $\vec{r}(t) = t^2\vec{i} + 2t\vec{j} + \ln t\,\vec{k} \quad 1 \le t \le e$

$\vec{r}\,'(t) = \langle 2t, 2, \frac{1}{t} \rangle \qquad |\vec{r}\,'(t)| = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = \sqrt{\left(\frac{1}{t} + 2t\right)^2}$

$= \frac{1}{|t|} + 2|t|$. Since $1 \le t \le e$, this is $\frac{1}{t} + 2t = |\vec{r}\,'(t)|$

$L = \int_1^e \left(\frac{1}{t} + 2t\right) dt = \ln t + t^2 \Big|_1^e = (1 - 0) + (e^2 - 1) = e^2$

16 $\vec{r}(t) = 12t\vec{i} + 8t^{3/2}\vec{j} + 3t^2\vec{k} \quad 0 \le t \le 1 \quad \vec{r}\,'(t) = \langle 12, 12t^{1/2}, 6t \rangle$

$|\vec{r}\,'(t)| = \sqrt{144 + 144t + 36t^2} = \sqrt{(6t + 12)^2} = |6t| + 12$

Since $0 \le t \le 1$, $|\vec{r}\,'(t)| = 6t + 12$

$L = \int_0^1 (6t + 12)\, dt = 3t^2 + 12t \Big|_0^1 = (3 - 0) + (12 - 0) = 15$