# LECTURE OUTLINE Velocity, acceleration, and curvature

**Professor Leibon** 

Math 8

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# Velocity Acceleration Curvature

Space Curve review

# Recall our Helix:

$$\vec{v}(t) = <\cos(t), \sin(t), t > 1$$

Sketch the helix. Find the velocity at each time of particle traveling along a helix. Watch is the direction of this velocity, this is usually denoted  $\hat{T}(t)$  and called the *unit tangent vector*. Find  $\frac{d}{dt}\hat{T}(t)$ 's direction vector for our helix, and call it  $\hat{N}(t)$  the *curve's normal vector*.

Recall from last time:  $\hat{T} \cdot \hat{N} = 0$ 

Let  $\hat{B}(t) = \hat{T}(t) \times \hat{N}(t)$ , this is usually called the curve's *binormal vector*. Express  $\frac{d}{dt}\hat{N}(t)$  as  $-a(t)\hat{T} + b(t)\hat{B}$  for our helix. Why is this always true?

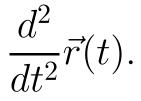
Compute  $\hat{B}$  and  $\frac{d}{dt}\hat{B}$  for our helix. Why must  $\frac{d}{dt}\hat{B}$  always be a multiple of  $\hat{N}$ ?

#### **Differentiation Rules**

- 1.  $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \frac{d\vec{u}}{dt}(t) + \frac{d\vec{v}}{dt}(t)$
- **2.**  $\frac{d}{dt}[c\vec{u}(t)] = c\frac{d\vec{u}}{dt}(t)$
- **3.**  $\frac{d}{dt}[f(t)\vec{u}(t)] = f(t)\frac{d\vec{u}}{dt}(t) + \frac{df}{dt}(t)\vec{u}$
- **4.**  $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \frac{d\vec{u}}{dt}(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \frac{d\vec{v}}{dt}(t)$
- **5.**  $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \frac{d\vec{u}}{dt}(t) \times \vec{v}(t) + \vec{u}(t) \times \frac{d\vec{v}}{dt}(t)$
- **6.**  $\frac{d}{dt}[\vec{u}(f(t))] = \frac{d\vec{u}}{dt}(t)\frac{df}{dt}(t)$

#### Acceleration

## The acceleration of our particle $\vec{r}(t)$ is



For  $t \geq a$ 

$$\frac{d}{dt}\vec{r}(t) = \int_{a}^{t} \frac{d^2}{dt^2}\vec{r}(t)dt + \frac{d}{dt}\vec{r}(a)$$

Find all curves that share our helix's acceleration vector.

**Emergency Slide:** Projectile Motion

Do in class....

## Arclength

$$s(t) = \int_{a}^{t} \left| \frac{d}{dt} \vec{r}(t) \right| dt$$

is the distance traveled as t went from a to t(watch out we are using the upper-limt = variable of integration!). Viewing a curve as parameterize by arclength means view is as a function of s.

View our helix as parameterized by arc length.

#### Curvature

# The curvature is $\kappa$ such that

$$\frac{d\hat{T}}{ds} = \kappa(s)\hat{N}$$

of

$$\frac{d\hat{T}}{dt}\frac{dt}{ds} = \left| \frac{d\hat{T}}{dt} \right| \left| \frac{dt}{ds} \right| = \frac{\left| \frac{d\hat{T}}{ds} \right|}{\left| \frac{d\hat{T}}{dt} \right|}$$

Find the curvature of our helix.

#### Frenet's Formulas

$$\frac{d\hat{T}}{ds} = \kappa \hat{N}$$
$$\frac{d\hat{N}}{ds} = -\kappa \hat{T} + \tau \hat{B}$$
$$\frac{d\hat{B}}{ds} = -\tau \hat{N}$$

 $\kappa$  is called the curve's *curvature* and  $\tau$  is called the curve's *torsion*.

Summarize of knowledge of the helix. Generalize to a general helix.

## An Example

Suppose we have a monstrous wheel of radius 1 meter, which we imagine rolling a rate of 1 revolution per second without slipping along the x-axis in the x,y-plane.

1. Find a formula for the position of a piece of gum attached to the circumference of the wheel which at time zero is on the wheel's bottom (this is curve traced out is called a *cycloid*).

- 2. Find our gum's velocity at each time.
- 3. Describe the curves that share our gum's velocity vector at each time.
- 4. Find the distance traversed by our gum at each time < 1/2 (why?).