LECTURE OUTLINE
Maxima and Minima

Professor Leibon

Math 8

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Review: Critical Points

We also call a point (a,b) a *critical point* of f if and only if $\nabla f(a,b) = 0$ or either f_x or f_y does not exist.

Fact: If f has a local maximum or minimum at (a,b) and the partial derivatives of f exist there, then $\nabla f = 0$.

Example: Find the critical points of $f(x,y) = x^2 + y^2 - 2x - 6y + 14$. Determine whether these points are minima or maxima.

Example: Find the critical points of $f(x,y) = y^2 - x^2$. Determine whether these points are minima or maxima.

Second Derivative Test

Second Derivative Test : Suppose (a, b) is a critical point of $f(x, y)$ and that the second partial derivatives of f are continuous on a disk with center (a, b). Let

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(a) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is not a local minimum or local maximum. (This is is called a *saddle point*.)

Example

Example 3: Apply the

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

test to our previous examples.

Example 4: Find the critical points of $f(x, y) = x^4 + y^4 - 4xy + 1$. Determine whether these points are minima, maxima, or saddles.

Extreme Values

Extreme Value Theorem: If f is a continuous on a closed, bounded set D in $\mathbf{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the absolute maximum and minimum values of a continuous function on a closed, bounded set D:

1. Find the value of f at each of the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum while the smallest is the absolute minimum.

Example

Example 5: Find the absolute maximum of $f(x, y) = x^2 + y^2 + x^2y + 4$ of the set $D = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$.

Example 2: Find the dimensions of the largest volume rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$.

The Wave Equation

The wave equation is

$$\frac{\partial^2 f}{\partial t^2} = a^2 \frac{\partial^2 f}{\partial x^2}.$$

Show that $f(x + at) + g(x - at)$ solves the wave equation.

Suppose you lift a sting in to the position given by e^{-x^2} and let it go. Explore what happens.