Homework due 11/17

1. $$\frac{\partial f}{\partial y}(x,y) = 3\cos(2x+3y)$$

$$\therefore \frac{\partial f}{\partial y}(-6,4) = 3\cos\left(2\cdot(-6)+3\cdot 4\right) = 3\cos(0) = 3$$

2. $$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\left((x+h)^2-(x+h)y+2y^2\right)-\left(x^2-xy+2y^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{x^2+2hx+h^2-xy-hy+2y^2-x^2+xy-2y^2}{h}$$

$$= \lim_{h\to 0} \frac{2hx+h^2-hy}{h}$$

$$= \lim_{h\to 0} (2x+h-y)$$

$$= 2x-y.$$

3. (a.) $g(y)$ does not depend on x [it is constant as a function of x]

so $$\frac{\partial z}{\partial x} = g(y)\cdot\frac{\partial f}{\partial x}(x) = g(y)\cdot f'(x)$$

In the same way, $$\frac{\partial z}{\partial y} = f(x)\cdot g'(y).$$

(b.) By the Chain Rule,

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}(xy) \cdot \frac{\partial (xy)}{\partial x} = \frac{\partial f}{\partial x}(xy) \cdot y = y f_x(xy)$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}(xy) \cdot \frac{\partial (xy)}{\partial y} = \frac{\partial f}{\partial x}(xy) \cdot x = x f_y(xy)$$

(c.) By the Chain Rule again,

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}(x/y) \cdot \frac{1}{y} = y^{-1} f_x(xy)$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}(x/y) \cdot (-x/y^2) = -xy^{-2} f_y(xy).$$

(4.) $f(x,y) = \ln(3x+5y)$

$$f_x(x,y) = \frac{3}{3x+5y}$$

$$f_{xx}(x,y) = \frac{-3 \cdot 3}{(3x+5y)^2} = \frac{-9}{(3x+5y)^2} \; ; \; f_{xy}(x,y) = \frac{(-1) \cdot 3 \cdot 5}{(3x+5y)^2} = \frac{-15}{(3x+5y)^2}$$

$$f_y(x,y) = \frac{5}{3x+5y}$$

$$f_{yx}(x,y) = \frac{-3 \cdot 5}{(3x+5y)^2} = \frac{-15}{(3x+5y)^2} \; ; \; f_{yy}(x,y) = \frac{-5 \cdot 5}{(3x+5y)^2} = \frac{-25}{(3x+5y)^2}$$

5. $z = u(v-w)^{1/2}$

$$\frac{\partial z}{\partial w} = -\frac{1}{2}u(v-w)^{-1/2}$$

$$\frac{\partial^2 z}{\partial v \partial w} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial w}\right) = \left(-\tfrac{1}{2}\right)^2 u(v-w)^{-3/2} = \frac{1}{4}u(v-w)^{-3/2}$$

$$\frac{\partial^3 z}{\partial u \partial v \partial w} = \frac{\partial}{\partial u}\left(\frac{\partial^2 z}{\partial v \partial w}\right) = \frac{1}{4}(v-w)^{-3/2}$$

6. (a) If we move from P in the positive x-direction the next level curve has a smaller value than $f(P)$, while in the negative x-direction the next level curve is greater than $f(P)$, so f_x is (probably!) negative at P.

(b) The function f is increasing as we go up in the y-direction, so f_y is (probably) positive at P.

(c) The level curve to the right of P is further parallel to the x-axis than the level curve to the left of P, i.e. the rate of decrease in f is slowing, so f is concave up, i.e. f_{xx} is positive.

(d) f_{xy} is the rate of change of f_x in the y-direction. ~~At~~ points above P the distance between level curves decreases so f is steeper, i.e. f_x decreases (because it is negative). Thus f_{xy} is negative.

(e.) The level curve above P is closer than the one below P, so f is increasing ever more rapidly in the y-direction, i.e. f is concave up, so f_{yy} is positive.

(7.) (a.) $\frac{\partial T}{\partial x}(x,y) = \frac{-2x \cdot 60}{(1+x^2+y^2)^2} = \frac{-120x}{(1+x^2+y^2)^2}$

so $\frac{\partial T}{\partial x}(2,1) = \frac{-120 \cdot 2}{(1+4+1)^2} = -\frac{40}{6} = -\frac{20}{3}$

(b.) $\frac{\partial T}{\partial y}(x,y) = \frac{-2y \cdot 60}{(1+x^2+y^2)^2} = \frac{-120y}{(1+x^2+y^2)^2}$

so $\frac{\partial T}{\partial y}(2,1) = \frac{-120}{6^2} = -\frac{10}{3}$

8. $\frac{\partial z}{\partial x}(4,1,0) = \frac{4}{1} = 4$

$\frac{\partial z}{\partial y}(4,1,0) = \ln 4.$

so the tangent plane at $(4,1,0)$ is

$$z - 0 = 4(x-4) + \ln 4(y-1)$$

i.e. $z = 4x + \ln 4\, y - 16 - \ln 4.$

9. $\frac{\partial z}{\partial x}(x,y) = 2x e^{x^2-y^2}$ $\qquad$ $\frac{\partial z}{\partial y}(x,y) = -2y\, e^{x^2-y^2}$

$\cancel{\partial z}$

The tangent plane at $(1,-1,1)$ is

$$z - 1 = \frac{\partial z}{\partial x}(1,-1)\cdot(x-1) + \frac{\partial z}{\partial y}(1,-1)\cdot(y-1)$$

$$z = 2(x-1) + 2(y-1) + 1$$

$$z = 2x + 2y - 3$$

10. $f_x(x,y) = \frac{1}{2}(x+e^{4y})^{-1/2}$

$f_y(x,y) = 4e^{4y}.\frac{1}{2}(x+e^{4y})^{-1/2} = 2e^{4y}(x+e^{4y})^{-1/2}$

$x+e^{4y} > 0$ for (x,y) near $(3,0)$, because $3+e^{4\cdot 0} = 4 > 0$. so f_x and f_y are continuous near 0. Thus f is differentiable at $(3,0)$.

The linearization is

$$L(x,y) = f(3,0) + f_x(3,0)\cdot(x-3) + f_y(3,0)\cdot(y-0)$$

$$= \sqrt{3+e^0} + \frac{1}{2\sqrt{3+e^0}}(x-3) + \frac{2\cdot e^0}{\sqrt{3+e^0}}\, y$$

$$= 2 + \frac{1}{4}(x-3) + 1\cdot y$$

$$= 5/4 + \frac{1}{4}x + y.$$

11. $\sin(2x+3y)$ is a composition of everywhere-differentiable functions, which is thus differentiable by the Chain Rule.

The linearization is

$$L(x,y) = \sin(2\cdot(-3)+3\cdot 2) + 2\cos(2\cdot(-3)+3\cdot 2)(x+3) + 3\sin(2\cdot(-3)+3\cdot 2)(y-2)$$

$$= 2(x+3)+3(y-2)$$

$$= 2x+3y.$$

Note: This is ~~the~~ a generalization of the single-variable result that for small x, $\sin x \approx x$.

12. $f_x(x,y) = \dfrac{-x}{\sqrt{20-x^2-7y^2}}$ $\qquad f_y(x,y) = \dfrac{-7y}{\sqrt{20-x^2-7y^2}}$

so $$L(x,y) = \sqrt{20-2^2-7\cdot 1^2} - \frac{2}{\sqrt{20-2^2-7}}(x-2) - \frac{7}{\sqrt{20-2^2-7}}(y-1)$$

$$= 3 - \frac{2}{3}(x-2) - \frac{7}{3}(y-1)$$

$$= \frac{20}{3} - \frac{2}{3}x - \frac{7}{3}y$$

Thus $f(1.95,1.08) \approx L(1.95,1.08) = \frac{20}{3} - \frac{2}{3}\cdot 1.95 - \frac{7}{3}\cdot 1.08$

$$= \frac{427}{150}$$

13. We must estimate $f_v(40,20)$ and $f_t(40,20)$ as we did on the last homework. Rather than repeat the tedious calculation, let me just tell you that my approximations are $f_v(40,20) \approx 1.15$, $f_t(40,20) \approx 0.45$.

Yours should be similar.

Then the linearization is

$$L(x,y) = f(40,20) + 1.15(v-40) + 0.45(t-20)$$

$$= 28 + 1.15v - 46 + 0.45t - 9$$

$$= 1.15v + 0.45t - 27$$

Thus the wave height at $(43,24)$ is

$$f(43,24) \approx L(43,24) = 1.15 \cdot 43 + 0.45 \cdot 24 - 27$$

$$= 33.25$$

14. $$\frac{\partial v}{\partial x} = y \cdot y \cdot (-\sin(xy)) = -y^2 \sin(xy)$$

$$\frac{\partial v}{\partial y} = \cos(xy) + y \cdot x \cdot (-\sin(xy)) = \cos(xy) - xy\sin(xy)$$

so $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -y^2\sin(xy)dx + (\cos(xy) - xy\sin(xy))dy$

15. $z = \sqrt{x^2+y^2}$, $x = e^{2t}$, $y = e^{-2t}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{x}{\sqrt{x^2+y^2}} \cdot 2e^{2t} + \frac{y}{\sqrt{x^2+y^2}} \cdot (-2e^{-2t})$$

$$= \frac{e^{2t}}{\sqrt{e^{4t}+e^{-4t}}} \cdot 2e^{2t} + \frac{e^{-2t}}{\sqrt{e^{4t}+e^{-4t}}} \cdot (-2e^{-2t})$$

$$= \frac{2(e^{4t}-e^{-4t})}{\sqrt{e^{4t}+e^{-4t}}}$$

16. $\omega = xy + yz^2$ $\quad x = e^t$ $\quad y = e^t \sin t$ $\quad z = e^t \cos t$

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial \omega}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial \omega}{\partial z} \cdot \frac{dz}{dt}$$

$$= y \cdot e^t + (x+z^2) \cdot (e^t \sin t + e^t \cos t) + 2yz(e^t \cos t - e^t \sin t)$$

$$\cancel{= (e^t \sin t) e^t + (e^t + e^{2t}\cos^2 t)(e^t \sin t + e^t \cos t) + 2e^t \sin t \cdot e^t \cos t (e^t \cos t - e^t \sin t)}$$

$$\cancel{= e^{2t}(\sin t + \sin t + \cos t + e^t \cos^2 t \sin t + e^t \cos^3 t + 2e^t \sin t \cos^2 t - 2e^t \sin^2 t \cos t)}$$

$$= e^t(y + (x+z^2)(\sin t + \cos t) + 2yz(\cos t - \sin t))$$

You may substitute for x, y and z in your answer if you want.