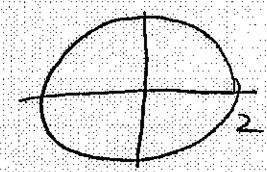
1. Find the maximum and minimum of $f(x, y) = x^2 + 2x + y^2$ on the disk $x^2 + y^2 \leq 4$.



1) Critical points inside disk:

$f_x = 2x+2 = 0$ $\Rightarrow$ $x = -1$ $y = 0$

$f_y = 2y = 0$ (which is inside disk)

$(-1, 0)$ $f(-1,0) = -1$

2) bdry.

On bdry $x^2+y^2 = 4$ so $f(x,y) = 4+2x$

As go around circle, x varies between -2 and 2. $4+2x$ varies between 0 and 8

~~4 and 12~~

min on bdry : 0

max on bdry : 8

3) Global max $= 8$

Global min $= -1$

2. Find the maximum and minimum of $T(x, y, z) = xyz$ subject to the constraint $x^2 + y^2 + 4z^2 = 12$. Note: It is easy to see that all absolute extrema occur when none of x, y, z are zero, so you may assume that fact in your solution.

Using Lagrange mults.

constraint $g(x,y,z) = x^2+y^2+4z^2=12$

$T_x = \lambda g_x \qquad yz = \lambda(2x) \qquad (i)$

$T_y = \lambda g_y \qquad xz = \lambda(2y) \qquad (ii)$

$T_z = \lambda g_z \qquad xy = \lambda(8z) \qquad (iii)$

$x^2+y^2+4z^2=12 \qquad (iv)$

constraint

$$\left.\begin{array}{ll} x\,(i) & xyz = \lambda(2x^2) \\ y\,(ii) & xyz = \lambda(2y^2) \\ z\,(iii) & xyz = \lambda(8z^2) \end{array}\right\} \Rightarrow x^2 = y^2 = 4z^2$$

Plug into constraint: $x^2 + \underbrace{x^2}_{y^2} + \underbrace{x^2}_{4z^2} = 12$

$x^2 = 4 \quad x = \pm 2 \qquad y = \pm 2 \qquad 4z^2 = 4,\ z = \pm 1$

max $(2)(2)(1) = 4$

min $-(2)(2)(1) = -4$

3. Let $f(x,y) = x^3y^4$.

(a) Find $\nabla f(1,1)$.

$\nabla f = \langle 3x^2y^4, \ 4x^3y^3 \rangle$

$\nabla f(1,1) = \langle 3, 4 \rangle$

(b) Find an equation of the tangent plane to the graph of f at the point $(1, 1, 1)$.

$f(1, 1) = 1$

$z - 1 = 3(x-1) + 4(y-1)$

(c) Find the maximum rate of increase of f at $(1, 1)$, and the direction in which it occurs.

$\max = |\nabla f(1,1)| = 5$

direction = direction of $\nabla f(1,1)$

$= \langle 3, 4 \rangle$

(or unit vector $\langle \frac{3}{5}, \frac{4}{5} \rangle$)

(d) A unit vector $\mathbf{u}$ makes an angle of $\pi/3$ with $\nabla f(1,1)$. Find the directional derivative $D_{\mathbf{u}}f(1,1)$. Hint: you don't really need to know $\mathbf{u}$ to answer this question.

$$
\begin{aligned}
D_{\vec{u}} f(1,1) &= \nabla f(1,1) \cdot \vec{u} \\
&= |\nabla f(1,1)| \underbrace{|\vec{u}|}_{1} \cos\theta \\
&= 5(1)\cos\theta \\
&= 5\left(\tfrac{1}{2}\right) = \tfrac{5}{2}
\end{aligned}
$$

(e) Determine an equation for the tangent line to the level curve of f at the point $(1, 1)$.

normal vector $\nabla f(1,1) = \langle 3, 4 \rangle$

$$3(x-1) + 4(y-1) = 0$$

4. Consider the following series. If they converge, determine their value; if they diverge, briefly say why.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

Converges to ______,

or

Diverges because $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$ (Test for divergence)

(b) $\displaystyle\sum_{n=0}^{\infty} 10^{10}\left(\frac{2}{3}\right)^{n}$

Converges to $10^{10} \frac{1}{1-\frac{2}{3}} = 3 \cdot 10^{10}$ (geom series),

or

Diverges because ______

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$.

Converges to e^2 since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$,

or

Diverges because ______

5. Below is drawn a contour map (the level curves) for a twice differentiable function $f(x, y)$. Use this map to answer the questions which follow:

   (a) Give the approximate coordinates (integer values) of a local maximum point of f.

   $(1, -1)$

(b) Give the approximate coordinates (integer values) of a saddle point of f.

$(-1, 1)$

(c) Consider the point $(0,0)$ marked on the diagram by a ■. You may assume that none of the partial derivatives of f are zero at this point. Indicate whether the following partial derivatives are positive or negative:

$f_x(0,0)$ is pos (f increasing as go right)

$f_y(0,0)$ is neg (f decreasing as go up)

6. Suppose a twice differentiable function f has a critical point at (x_0, y_0). In each of the following, information about the second order partials is given. In each case, classify the critical point as a local maximum, local minimum, or saddle point, or else explain why the second derivative test fails.

   (a) $f_{xx}(x_0, y_0) = 2, f_{yy}(x_0, y_0) = 6, f_{xy}(x_0, y_0) = 2.$

$f_{xx} f_{yy} - f_{xy}^2 = (2)(6) - 4 > 0$

$f_{xx} > 0$ local min

(b) $f_{xx}(x_0, y_0) = 2, f_{yy}(x_0, y_0) = 8, f_{xy}(x_0, y_0) = 4.$

$$f_{xx} f_{yy} - f_{xy}^2 = 2(8) - 16 = 0$$

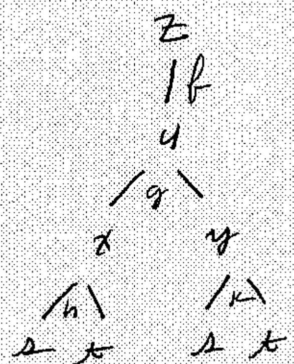
test fails

(c) $f_{xx}(x_0, y_0) = 2, f_{yy}(x_0, y_0) = 6, f_{xy}(x_0, y_0) = 5.$

$f_{xx}f_{yy} - f_{xy}^2 = (2)(6) - 25 < 0$

saddle pt.

7. Given a function $z = f(u)$ where $u = g(x, y)$, $x = h(s, t)$, and $y = k(s, t)$, use the Chain Rule to write an expression for $\dfrac{\partial z}{\partial t}$ in terms of the partial derivatives of the other functions.



$$\frac{\partial z}{\partial t} = \frac{dz}{du}\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{dz}{du}\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$= f'(u)\, g_x\, h_t + f'(u)\, g_y\, k_t$$

8. **Multiple Choice** Circle the correct response.

A. What is the Maclaurin series for $\dfrac{1}{1+x^2}$?

a. $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$

b. $\displaystyle\sum_{n=0}^{\infty} x^{2n}$

c. $\displaystyle\sum_{n=0}^{\infty}(-1)^n (2n) x^{2n-1}$

d. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{x^{2n-1}}{2n}$

e. none of the above

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

B. If the motion of a particle is given by a vector-valued function $\mathbf{r}(t)$ defined for $a \leq t \leq b$, then the integral of the speed of $\mathbf{r}(t)$ from $t = a$ to $t = b$ equals

a. the acceleration

b. the distance from $\mathbf{r}(a)$ to $\mathbf{r}(b)$

c. the velocity

d. the distance the particle travels going from $\mathbf{r}(a)$ to $\mathbf{r}(b)$

e. none of the above

$\int_a^b |\vec{r}\,'(t)|\,dt$ = distance along curve from $\vec{r}(a)$ to $\vec{r}(b)$ (arclength)

C . Which of the following vectors is orthogonal to the plane containing the parallel lines $\mathbf{r}(t) = \langle 1,1,1\rangle + t\langle 2,1,4\rangle$, and $\mathbf{r}(t) = \langle 2,3,4\rangle + t\langle 2,1,4\rangle$.

a. $\langle 1,1,1\rangle \times \langle 2,3,4\rangle$

b. $\langle 1,1,1\rangle \times \langle 2,1,4\rangle$

c. $\langle 2,1,4\rangle$

d. $\langle 1,2,3\rangle \times \langle 2,1,4\rangle$

e. $\langle 1,2,3\rangle$

9. Find

$$\int \frac{1}{x^2\sqrt{x^2+9}}dx.$$

Subst $x = 3\tan\theta \quad dx = 3\sec^2\theta\, d\theta$

$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = \sqrt{9\sec^2\theta} = 3\sec\theta$

$$\int \frac{3\sec^2\theta}{9\tan^2\theta(3\sec(\theta))}d\theta$$

$$= \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

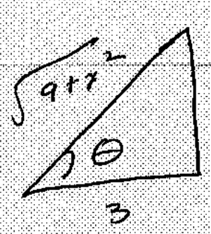
$$= \frac{1}{9}\int \frac{1}{\cos\theta}\,\frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$\left(= \frac{1}{9}\int \frac{du}{u^2} \text{ where } u = \sin\theta\right)$

$$= -\frac{1}{9\sin\theta} + C$$

$$= -\frac{1}{9}\frac{\sqrt{9+x^2}}{x} + C$$



$\tan\theta = \frac{x}{3}$

$\sin\theta = \frac{x}{\sqrt{9+x^2}}$

10. Determine whether the planes given by $x + 4y - 3z = 1$ and $-3x + 6y + 7z = 3$ are parallel, perpendicular, or neither. If neither, find the angle between them.

normal vectors $\langle 1, 4, -3 \rangle$ $\vec{n}_1$

$\langle -3, 6, 7 \rangle$ $\vec{n}_2$

not parallel

$\langle 1, 4, -3 \rangle \cdot \langle -3, 6, 7 \rangle = -3 + 24 - 21 = 0$

perpendicular