Fast & Dirty Solutions

Be Warned!

MATH 8: Practice Midterm II
November 10, 2004

Show all your work. Full credit may not be given for correct answers if they are not adequately justified. Good luck!

1. Find a power series that converges to each of the following functions and give the radius and interval of convergence. (Do this by manipulating geometric series, **not** by Taylor's formula.)

   (a) $\ln(1+x)$.

$$\ln(1+x) + C = \int \frac{1}{1+x}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-x)^n \, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{n+1}}{n+1}$$

& at $x=0$

$$\ln(1+0) + C = 0 + C = \sum \frac{(-0)^{n+1}}{n+1}$$

so $C = 0$ &

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

To find the radius of convergence we use the Ratio Test, hence need

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\frac{x^{n+1}}{x}\right|$$

$$= |x| \lim_{n\to\infty}\left|1+\frac{1}{n+1}\right| = |x| < 1$$

which shows us $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}$ has radius of convergence equal to $\boxed{1}$.

Now check the end points

at $x=1$ $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}1^n}{n}$ is alternating with

$\lim_{n\to\infty}\frac{1}{n}=0$ & $\frac{1}{n+1}<\frac{1}{n}$, so by the Alternating Series test $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges.

at $x=-1$ we have $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)^n}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$

wich is harmonic, hence diverges.

So the Interval of Convergence is

$$\boxed{(-1, 1]}$$

(b) $\frac{3}{27-x^3}$.

$$\frac{3}{27-x^3} = \frac{3}{27}\left(\frac{1}{1-\frac{x^3}{27}}\right) = \frac{1}{9}\left(\frac{1}{1-\left(\frac{x}{3}\right)^3}\right)$$

$$= \frac{1}{9}\sum_{n=0}^{\infty}\left(\left(\frac{x}{3}\right)^3\right)^n$$ with interval of Convergence $\left|\frac{x}{3}\right| < 1$, since this the geometric series.

In other words,

$$\boxed{\frac{2}{27-x^3} = \sum_{n=0}^{\infty} \frac{1}{3^{3n+2}} x^{3n} \text{ for } x \text{ in } (-3, 3).}$$

the interval of Convergence

&

$$\boxed{\text{Radius of Convergence} = 3}$$

2. Suppose you have a function $f(x)$ such that $f(x)$'s third Taylor polynomial at $x = 1$ is $P_3(x) = 1 - (1/2)(x-1) + (x-1)^2 + (2/3)(x-1)^3$, and assume that all of $f(x)$'s derivatives satisfy $\left|\frac{d^n f}{dx^n}\right| \leq 5$ on the interval $(0, 2)$.

(a) Given the above data, approximate $f(1.5)$.

$$f(1.5) = f\left(\frac{3}{2}\right) \approx 1 - \frac{1}{2}\left(\frac{3}{2}-1\right) + \left(\frac{3}{2}-1\right)^2 + \left(\frac{2}{3}\right)\left(\frac{3}{2}-1\right)^3$$

$$\approx 1 - \frac{1}{4} + \frac{1}{4} + \frac{1}{12} = \boxed{\frac{13}{12}}$$

(b) Bound the difference $|f(1.5) - P_3(1.5)|$ using the above data, and justify your answer.

By Taylor Remainder Estimate,

$$\left|f\left(\tfrac{3}{2}\right) - P_3\left(\tfrac{3}{2}\right)\right| \leq \frac{M\left(\frac{3}{2}-1\right)^4}{4!}$$

& $M = \left|\max f^{4}(x) \text{ on } [1, 1.5]\right| \leq 5$

so

$$\boxed{\left|f\left(\tfrac{3}{2}\right) - P_3\left(\tfrac{3}{2}\right)\right| \leq \frac{5}{2^4\, 4!} = \frac{5}{3}\,\frac{1}{2^7}} \leq \frac{1}{2^6} = \frac{1}{64}$$

(c) Given the above data, can you determine $f(x)$'s second derivative at $x = 1$? If so find it, if not why.

$$P_3(x) = \sum_{n=0}^{3} \frac{f^n(1)}{n!}(x-1)^n$$

So

$$\frac{f^2(1)}{2!} = 1 \quad \text{from}$$

$$P_2(x) = 1 - (\tfrac{1}{2})(x-1) + (x-1)^2$$

&

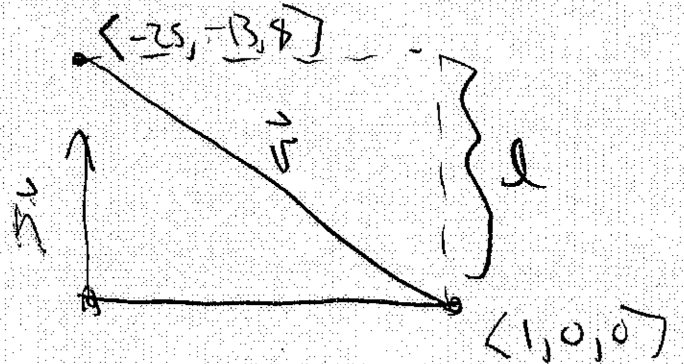
$$f^2(1) = \frac{d^2 f}{dx^2}(1) = (2!)\cdot 1 = 2$$

3. Find the distance between the point $P = (-25, -13, 8)$ and the plane with equation $3x + y - z = 3$.

$\vec{n} = \langle 3, 1, 1 \rangle$ is nomal to the plane

& $\langle 1, 0, 0 \rangle$ is on the plance

So we need l in



$$l = \left| \vec{v} \cdot \frac{\vec{n}}{|\vec{n}|} \right| = \left| \langle -26, -13, 8 \rangle \cdot \frac{\langle 3, 1, 1 \rangle}{\sqrt{11}} \right|$$

$$= \left| -\frac{99}{\sqrt{11}} \right| = \boxed{9\sqrt{11}}$$

4. Find the line of intersection of the planes $x+y+z=3$ and $x+2y+3z=6$.

Find $\vec{v}$ & $\vec{p}$ in $\vec{r}(t) = t\vec{v} + \vec{p}$.

Well $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

$= \hat{i} - 2\hat{j} + \hat{k}$

& assume $x=0$ a point on both planes satisfies

$0 + y + z = 3$ so $z = 3 - y$

& $0 + 2y + 3z = 6$ $\quad 2y + 9 - 3y = 6$

$-y = -3$

Hence $y = 3$ & $z = 0$

so $\vec{p} = \langle 0, 3, 0 \rangle$ is on both planes

&

$\vec{r}(t) = \langle t, -2t+3, t \rangle$ is our needed line.

5. Suppose $\vec{u}$ and $\vec{v}$ are in the plane containing the origin determined by $3x+2y+z=0$ and that that $\vec{u}$, $\vec{v}$, and $\vec{w}$ satisfy $\vec{v}\cdot(\vec{w}\times\vec{u})=0$. What is the equation of a plane through the origin that contians $\vec{u}$ and $\vec{w}$? Why?

Well

$$|\vec{v}\cdot(\vec{w}\times\vec{u})| = \left|\text{volume parallellopiped spanned by } \vec{u},\vec{v},\vec{w}\right|$$

$$= 0$$

So the parallellopiped is flat

& $\vec{u}, \vec{v}, \vec{w}$ are all in the

$\boxed{\text{same}}$ plane.

Hence the equation of this plane is

$$\boxed{3x+2y+z=0}$$

6. Suppose $\vec{u}(3) = < 1, 1, 2 >$, $\vec{v}(3) = < 3, 1, -1 >$, $\frac{d\vec{u}}{dt}(3) = < -1, 0, 2 >$ and $\frac{d\vec{v}}{dt}(3) = < 0, -2, 3 >$.

(a) Compute $\frac{d}{dt}[\vec{u} \cdot \vec{v}]$ at $t = 3$.

$$\frac{d}{dt}[\vec{u} \cdot \vec{v}]\Big|_{t=3} = \frac{d\vec{u}}{dt} \cdot \vec{v} + \vec{u} \cdot \frac{d\vec{v}}{dt}\Big|_{t=3}$$

$$= \langle -1, 0, 2\rangle \cdot \langle 3, 1, -1\rangle + \langle 1, 1, 2\rangle \cdot \langle 0, -2, 3\rangle$$

$$= -3 - 2 - 2 + 6 = -7 + 6 = \boxed{-1}$$

(b) Compute $\frac{d}{dt}[\vec{u} \times \vec{v}]$ at $t = 3$.

$$\frac{d}{dt}\left[\vec{u} \times \vec{v}\right]\Big|_{t=3} = \frac{d\vec{u}}{dt} \times \vec{v} + \vec{u} \times \frac{d\vec{v}}{dt}\Big|_{t=3}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 2 \\ 3 & 1 & -1 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 0 & -2 & 3 \end{vmatrix}$$

$$= (-2, 5, -1) + (7, -3, -2)$$

$$= \boxed{(5, 2, -3)}$$

(c) Compute $\frac{d}{dt}[e^t\vec{u}]$ at $t=3$.

$$\frac{d}{dt}\left[e^t\vec{u}\right]\Big|_{t=3} = \left(\frac{de^t}{dt}\right)\vec{u} + e^t\frac{d\vec{u}}{dt}\Big|_{t=3}$$

$$= e^t\vec{u} + e^t\frac{d\vec{u}}{dt}\Big|_{t=3}$$

$$= e^3(1,1,2) + e^3(-1,0,2)$$

$$= \boxed{(0, e^3, 4e^3)}$$

7. Let $\vec{r}(t) = < \sin(t) + t, \cos(t), 3 >$.

(a) Find the tangent line to the curve given by $\vec{r}(t)$ at $t = \frac{\pi}{4}$.

We need $\vec{v}$ & $\vec{p}$ in $\vec{l}(t) = t\vec{v} + \vec{p}$

$$\vec{v} = \left.\frac{d\vec{r}}{dt}\right|_{t=\frac{\pi}{4}} = \left.\left(\cos(t)+1, -\sin(t), 0\right)\right|_{t=\frac{\pi}{4}}$$

$$= \left(\frac{\sqrt{2}}{2}+1, -\frac{\sqrt{2}}{2}, 0\right)$$

$$\vec{p} = \left.\left(\sin(t)+t, \cos(t), 3\right)\right|_{t=\frac{\pi}{2}}$$

$$= \left(\frac{\sqrt{2}}{2}+\frac{\pi}{4}, \frac{\sqrt{2}}{2}, 3\right)$$

$$\boxed{\vec{l}(t) = \left(\left(\frac{\sqrt{2}}{2}+1\right)t + \left(\frac{\sqrt{2}}{2}+\frac{\pi}{4}\right), -\frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}, 3\right)}$$

(b) Find the legnth of this curve for $0 \le t \le 1$. (Hint: $1 + \cos(2\theta) = 2\cos(\theta)^2$).

$$L = \int_0^1 \left| \frac{dr}{dt} \right| dt = \int_0^1 \sqrt{(\cos(t)+1)^2 + (\sin(t))^2}\, dt$$

$$= \int_0^1 \sqrt{(\cos(t))^2 + (\sin(t))^2 + 2\cos(t) + 1}\ dt$$

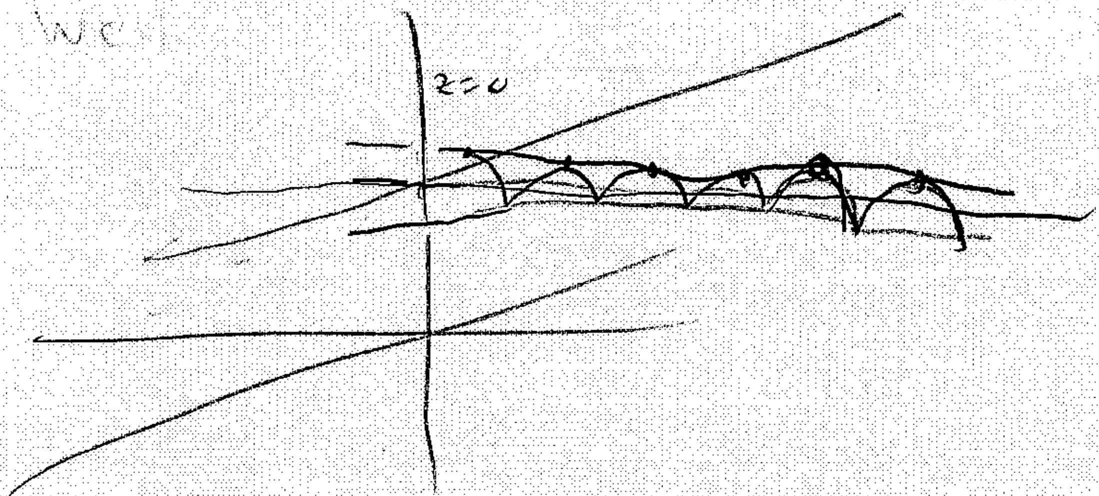
$$= \int_0^1 \sqrt{2 + 2\cos(t)}\, dt = \int_0^1 \sqrt{4\left(\cos\left(\tfrac{t}{2}\right)\right)^2}\, dt$$

$$= 2\int_0^1 \sqrt{\cos\left(\tfrac{t}{2}\right)^2}\, dt = 2\int_0^1 \cos\left(\tfrac{t}{2}\right) dt$$

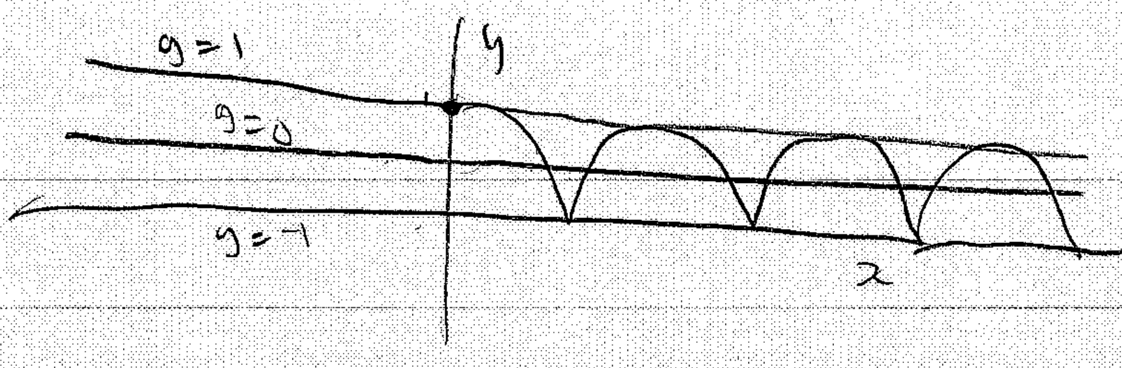
(Since $\cos\left(\frac{t}{2}\right) \ge 0$ for $0 \le t \le 1$)

$$= 4\sin\left(\tfrac{t}{2}\right)\Big|_0^1 = \boxed{4\sin\left(\tfrac{1}{2}\right)}$$

(c) Sketch the curve (Challenging).



from above



8. Find the third degree Taylor polynomial for $\tan(x)$ about $a = \frac{\pi}{4}$.

$$f(x)\Big|_{x=\pi/4} = \tan(x)\Big|_{x=\pi/4} = 1$$

$$f'(x)\Big|_{x=\pi/4} = 1+(\tan(x))^2\Big|_{x=\pi/4} = 2$$

$$f^2(x)\Big|_{x=\pi/4} = 2\tan(x)\left(1+(\tan(x))^2\right)\Big|_{x=\pi/4} = 4$$

$$f^3(x)\Big|_{x=\pi/4} = 2\left(1+(\tan(x))^2\right)\left(1+3(\tan(x))^2\right) = 16$$

So the need polynomial is

$$1 + 2\left(x-\frac{\pi}{4}\right) + \frac{4}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x-\frac{\pi}{4}\right)^3$$

$$= 1 + 2\left(x-\frac{\pi}{4}\right) + 2\left(x-\frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x-\frac{\pi}{4}\right)^3$$

9. Suppose we have a plane containing the points $(1, 1, 0)$, $(2, 1, 3)$ and $(1, 0, 5)$, and a line determined by $\frac{x-2}{2} = \frac{y-3}{5} = z - 1$.

(a) Find an equation for the plane.

Need $\vec{n}$ & $\vec{p}$ in $\vec{n}(\vec{r}-p) = 0$

well $(1,0,5) - (1,1,0) = (0,-1,5)$

&

$(2,1,3) - (1,1,0) = (1,0,3)$

are in the plane so

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 5 \\ 1 & 0 & 3 \end{vmatrix} = (-3, 5, 1)$$

$\vec{p} = (1,1,0)$ will work

& we have

$$\vec{n}\cdot(\vec{r}-\vec{p}) = \boxed{-3(x-1) + 5(y-1) + z = 0}$$

-10

(b) Do the plane and line intersect? If so find the points of intersection.

We can express our line in parametric form as

$\langle 2t+2, 5t+3, t+1 \rangle$

& we need a solution to our plane's eqn.

$-3(2t+2+1) + 5(5t+3-1) + (t+1) = 0$

or

$(-6+25+1)t + (-3+10+1) = 0$

or $t = \frac{-8}{20} = \frac{-2}{5}$ &

$$\boxed{\left\langle 2\left(\tfrac{-2}{5}\right)+2,\ 5\left(\tfrac{-2}{5}\right)+3,\ \tfrac{-2}{5}+1 \right\rangle = \left\langle \tfrac{6}{5}, 1, \tfrac{3}{5} \right\rangle}$$

is on the plane & on the line.

10. Find the vector projection and the scalar projection (i.e., component) of $\vec{b}$ on $\vec{a}$ where $\vec{b} = < 2, 1, 4 >$ and $\vec{a} = < 1, 2, 3 >$.

Scalar Projection of $\vec{b}$ on $\vec{a}$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{\langle 1,2,3 \rangle \cdot \langle 2,1,4 \rangle}{|\langle 1,2,3 \rangle|}$$

$$= \frac{16}{\sqrt{1^2+2^2+3^2}} = \boxed{\frac{16}{\sqrt{14}}}$$

Vector Projection of $\vec{b}$ on $\vec{a}$ is

$$(\text{Scalar Projection}) \cdot \frac{\vec{a}}{|a|}$$

$$= \frac{16}{\sqrt{14}} \frac{\langle 1,2,3 \rangle}{\sqrt{14}} = \boxed{\left\langle \frac{8}{7}, \frac{16}{7}, \frac{24}{7} \right\rangle}$$