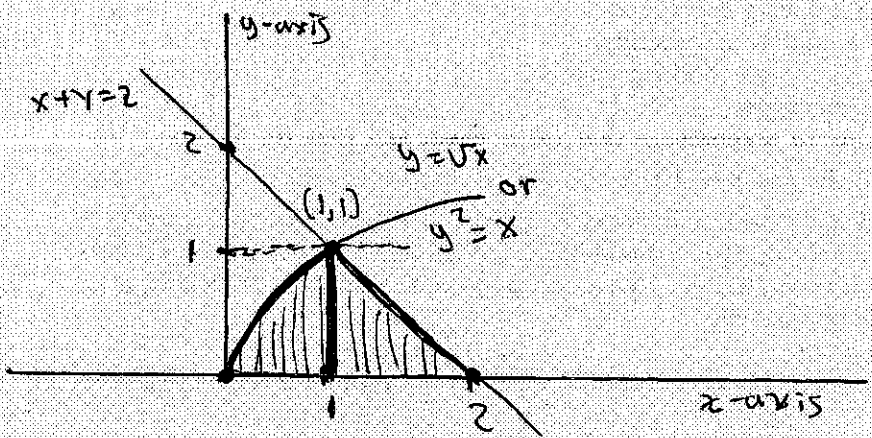
Numbers 2 and 7(c)

Warning To receive full credit on either of these problems did **not** require being as thorough as outlined. However, if any key step was missing or not justified in a way that the grader found clear, then points were taken off. The best thing to do is to be **careful**.

2. (10 points) Find the area bounded by the curve $y = \sqrt{x}$, the line $x + y = 2$ and the x-axis.

The key to this problem is getting a good picture.



Method 1: y-axis view

Viewing this problem from the point of view of the y-axis, we see that this region's area is the area between the curves $x = y^2$ and $x = 2 - y$ as y ranges from 0 to 1. Since $2 - y > y^2$ for y in $[0, 1]$, this area equals

$$\int_0^1 (2-y) - y^2 dy = \frac{-y^3}{3} - \frac{y^2}{2} + 2y\Big|_0^1 = 2 - \frac{1}{3} - \frac{1}{2} = \frac{7}{6}.$$

Method 2: x-axis view

The are of this curve is the area of the pictured triangle $\frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$ plus under the curve bounded by $y = \sqrt{x}$, the x-axis and the line $x = 1$. In other words

$$\frac{1}{2} + \int_0^1 \sqrt{x}dx = \frac{1}{2} + \left(\frac{2}{3}x^{3/2}\Big|_0^1\right) = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}.$$

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7. (7+7+8 points) Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \ln(\frac{n}{n+1})$. (Hint: You can compute the partial sums explicitly.)

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} [\ln n - \ln(n+1)]$$

n^{th} partial sum:

$$S_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \dots + (\ln(n) - \ln(n+1))$$

(telescopes)

$$= \ln 1 - \ln(n+1) \quad \text{(all else cancels)}$$

$$= 0 - \ln(n+1)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\ln(n+1) = -\infty$$

so series diverges.

7 (c). (8 points) Determine whether the following series converge or diverge: $\sum \frac{\sqrt{n^7+n^2}}{n^5+9}$.

Initial Glance: What is our series behaving like? Well notice

$$a_n = \frac{\sqrt{n^7+n^2}}{n^5+9} = \frac{\sqrt{n^7}\sqrt{1+1/n^5}}{n^5(1+9/n^5)} = \left(\frac{1}{n^{3/2}}\right)\left(\frac{\sqrt{1+1/n^5}}{1+9/n^5}\right). \tag{1}$$

Now we must decide on how to use this information, and it appears some sort of comparison to $\sum b_n$ where $b_n = \frac{1}{n^{3/2}}$ is in order.

Method 1, The Comparison Test

To use the **limit comparison test** to test that $\sum a_n$ converges will require checking the following hypothesi:

(a) The a_n and b_n are positive.

(b) $\lim_{n\to\infty} \frac{a_n}{b_n}$ exist and equals $c > 0$.

(c) $\sum b_n$ converges.

One **needs** to acknowledge that (a) is clear, and that (c) follows by the $p-test$ and the fact $p = \frac{3}{2} > 1$. To see (b) first we let $f(x) = \sqrt{1+\frac{1}{x^5}}$ and $g(x) = 1+\frac{9}{x^5}$ and note from basic calculus, $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to\infty} g(x) = 1$, and hence $\lim_{n\to\infty} f(n) = 1$ and $\lim_{n\to\infty} g(n) = 1$. Now our basic limit laws assure us that ratio $f(n)/g(n)$ converges to $1/1 = 1$. Using this, (b) follows from equation (1) by observing

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n^5}}}{1+\frac{9}{n^5}} = \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

Method 2, The Comparison Test

To use the **comparison test** to test that $\sum a_n$ converges will require checking the following hypothesi:

(a) The a_n and b_n are positive.

(b) $a_n \le b_n$

(c) $\sum b_n$ converges.

One **needs** to acknowledge that (a) is clear, and that (c) follows by the $p-test$ and the fact $\frac{3}{2} > 1$. One way to check (b) is a **well justified** version of the following observation

$$\frac{\sqrt{1+1/n^5}}{1+9/n^5} < \frac{\sqrt{1+9/n^5}}{1+9/n^5} = \frac{1}{\sqrt{1+9/n^5}} < 1,$$

Justification: well if $x > y > 0$, then $\sqrt{x} > \sqrt{y}$ since $\sqrt{x}$ is an increasing function. Hence the first inequality follows from the fact we increased the numerator and the last from the fact that we decreased the denominator (of a ratio of positive number.) From this, (b) follows by observing

$$a_n = \left(\frac{1}{n^{3/2}}\right)\left(\frac{\sqrt{1+1/n^5}}{1+9/n^5}\right) < \left(\frac{1}{n^{3/2}}\right)(1) = b_n.$$

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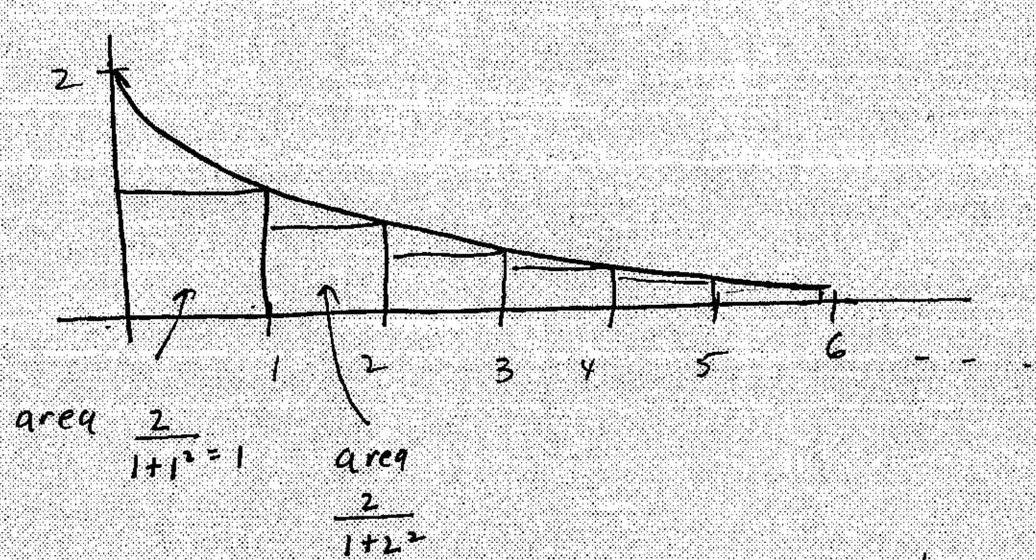
8. (10 points) Which is larger: $\displaystyle\sum_{n=1}^{\infty} \frac{2}{1+n^2}$ or π ?

(You must explain your answer fully and include a sketch. No credit will be given for an answer that is not explained. Hint: Integral test.)

Let $f(x) = \dfrac{2}{1+x^2}$ a positive, cts., decreasing function.



The sum of the series is the sum of the areas of the rectangles drawn above. The fastest way to answer this question is to note from the sketch that

$$\sum_{n=1}^{\infty} \frac{2}{1+n^2} < \int_0^{\infty} \frac{2}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{2}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} 2\tan^{-1}x \Big|_0^t = \lim_{t\to\infty} \left(2\tan^{-1}t - 2\tan^{-1}0\right)$$

$$= 2\left(\frac{\pi}{2}\right) - 0 = \pi.$$

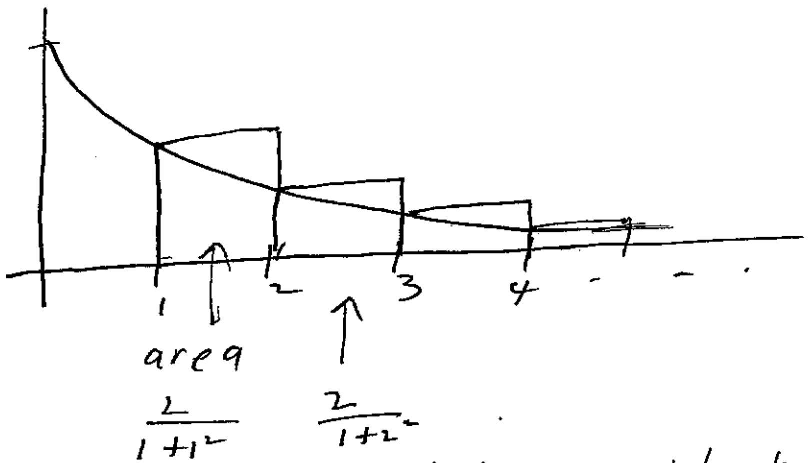
Or you could use sketch to say $\displaystyle\sum_{n=1}^{\infty} \frac{2}{1+n^2} < 1 + \int_1^{\infty} \frac{2}{1+x^2}\,dx$

$$= \dots = 1 + \frac{\pi}{2} < \pi$$

Additional comments on problem 8:

Whenever you're comparing a series to an integral, you have a choice of drawing the rectangles to touch the curve in their upper right corners or upper left. You need to experiment to see what gives you the info you need in the particular problem.

In this case, suppose you tried drawing rectangles as follows:



Then you would see that

$$\sum_{n=1}^{\infty} \frac{2}{1+n^2} = \Sigma(\text{areas of rects}) > \underbrace{\text{area under curve from } 1 \text{ to } \infty}_{=\ \int_1^\infty \frac{1}{1+x^2}\,dx = \ldots = \frac{\pi}{2}}$$

But ~~this would~~ at this point you would have shown $\sum_1^\infty \frac{2}{1+n^2} > \frac{\pi}{2}$ but wouldn't know whether sum $> \pi$. That would ~~tell~~ give a message that you need to go back & draw the rectangles the other way.