Math 81. *Abstract Algebra.*

Homework 6. Due on Wednesday, 2/24/2010.

1. Pages 581-2, problems 3, 4, 14.

2. Let K/F be a finite separable extension.

   (a) Show that there is a "smallest" finite extension L of K with L/F Galois. L is called the Galois closure of K/F.

   (b) Determine the Galois closure L of $\mathbb{Q}(\sqrt[3]{2}, \sqrt[5]{2})/\mathbb{Q}$, and compute its degree over $\mathbb{Q}$.

   (c) For L as in the previous part, determine whether or not $\text{Gal}(L/Q)$ is abelian.

3. Suppose that K/F is a finite Galois extension of degree n with Galois group $G = \{\sigma_1, \ldots, \sigma_n\}$. For an element $\alpha \in K$, define its trace and norm as follows:

$$Tr(\alpha) = Tr_{K/F}(\alpha) = \sigma_1(\alpha) + \cdots + \sigma_n(\alpha), \qquad N(\alpha) = N_{K/F}(\alpha) = \sigma_1(\alpha) \ldots \sigma_n(\alpha).$$

   (a) Show that Tr and N map K to F, and satisfy $Tr(\alpha + \beta) = Tr(\alpha) + Tr(\beta)$ and $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in K$.

   (b) Show that Tr is a surjective map. *[Hint: first show that there is an element $\alpha \in K$ for which $Tr(\alpha)$ is not zero. Note that in characteristic 0 or characteristic p with p not dividing n, this is easy, but there is a general way to do this in all cases.]*