Math 81. *Abstract Algebra.*

Homework 1. Due on Wednesday, 1/13/2010.

This problem set is intended to remind you of material from Math 71. Below are a few handy results which you may use without proof, but if they are unfamiliar, you should read the relevant material in the text.

A polynomial $f \in \mathbb{Z}[x]$ is called *primitive* if the gcd of its coefficients is 1. The following two theorems are equivalent to Gauss' lemma over $\mathbb{Q}$.

Theorem: Let $f \in \mathbb{Z}[x]$. Then f is irreducible in $\mathbb{Z}[x]$ if and only if f is primitive in $\mathbb{Z}[x]$ and irreducible in $\mathbb{Q}[x]$.

Theorem: Let $f \in \mathbb{Z}[x]$, and suppose that $f = gh$, where $g, h \in \mathbb{Q}[x]$. Then $f = g_0 h_0$ for some $g_0, h_0 \in \mathbb{Z}[x]$ with $\deg(g) = \deg(g_0)$ and $\deg(h) = \deg(h_0)$. In particular g_0 and h_0 are rational scalar multiples of g and h respectively.

1. Show that there exist ring homomorphisms $\mathbb{Z}_m \to \mathbb{Z}_n$ if and only if $n|m$. (We assume that ring homomorphisms take the multiplicative identity to the multiplicative identity.) Show that all such homomorphisms must be surjective.

2. For each of the ideals I listed below, determine whether the ideal I is prime, maximal, or neither in each of $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ by examining the appropriate quotient ring. If the quotient is not an integral domain, find zero divisors. If the quotient is not a field, then I is not maximal, so find a maximal ideal M with $I \subsetneq M$.
   (a) $I = (x^3 + 2)$
   (b) $I = (5, x^3 + 2)$
   (c) $I = (7, x^3 + 2)$

3. Consider the ring $\mathbb{Z}[x]$.
   (a) Which (if any) of $(x^3 + 2, x^3 + 9)$ or $(x^3 + 2, x^3 + 7)$ are maximal ideals?
   (b) Find infinitely many maximal ideals containing $(x^3 + x^2)$.

4. Let f be a nonconstant polynomial in $\mathbb{Q}[x]$. Show that there are only finitely many maximal ideals in $\mathbb{Q}[x]$ containing (f).

5. Proof or counterexample: Let P be a nonzero prime ideal in $\mathbb{Z}[x]$, and I an ideal with $P \subseteq I \subsetneq \mathbb{Z}[x]$. Then I is a prime ideal.