The Galois Correspondence

Here we state the Galois correspondence for pointed covers. (I've decided to rephrase it slightly in order to have it better correspond to the Galois correspondence form field theory.) **Assume that every space discussed below will be a path connected, locally path connected, semi-locally simply connect topological space.** For most of the below discussion these condition can be relaxed but at first go it does not seem worth the effort.

First let us carefully define pointed maps and spaces.

Definition 1 A space Z together with a specified point $z \in Z$ will be called a pointed space and denoted (Z, z). A pointed map $f : (Z, z) \to (W, w)$ will be a continuous map such that $f(z) = w$.

When you have a pointed space and/or map you are free to "forget" the point. In what follows some concepts will defined in the setting of unpointed spaces and applied to the setting of pointed spaces by forgetting the point. For example the following concept will prove crucial in both settings....

Definition 2 Let Y be a cover of X via the covering map ρ_x^y. Define $Deck(Y \preceq X)$ as the subgroup of the group of homeomorphisms of Y such that each $\phi \in Deck(Y \preceq X)$ satisfies $\rho_x^y \circ \phi = \rho_x^y$. $Deck(Y \preceq X)$ is called the group of deck transformations, or the group of covering transformation, or the automorphism group of the cover.

We proved long ago that $Deck(Y \preceq X)$ is a group. With this concept we can define and explore the notion of a pointed cover.

Definition 3 Fix a pointed space (X, x) and let $\rho_x^y : (Y, y) \to (X, x)$ and $\rho_x^z : (Z, z) \to (X, x)$ be coverings of (X, x). We say that Y and Z are equivalent as covers if there is a homeomorphism $f : Y \to Z$ such that $\rho_x^z \circ f = \rho_x^y$. We will view (Y, y) and (Z, z) as equivalent if Y and Z are equivalent via a homeomorphism where $f(y) = z$. Denote the equivalence class of such pairs as $[Y, y]$ and call this equivalence class a pointed cover of (X, x). Once we fix (Y, y) we will view $[Y, y]$ as the set of elements in the form $(Y, \hat{y})$.

Problem 4 Show $[Y, y_1] = [Y, y_2]$ if and only if there is a $\phi \in Deck(Y \preceq X)$ such that $\phi(y_1) = y_2$. Fixing Y explain how the set $[Y, y]$ is naturally acted on by $Deck(Y \preceq X)$. Explain how to identify $Deck(Y \preceq X)$ with a subgroup of the group of permutations of $(\rho_x^y)^{-1}(x)$.

Now we articulate the notion of a completely symmetric cover.

Definition 5 We call a cover normal if for every $y_1, y_2 \in (\rho^y)^{-1}(x)$ there is a $\phi \in Deck(Y \preceq X)$ such that $\phi(y_1) = y_2$.

Notice that if Y is a normal cover then there is only one equivalence class in the form $[Y, y]$. Letting $\equiv$ denote a set bijection, we see that $[Y, y] \equiv Deck[Y \preceq X]$.

The Galois correspondence is between subgroups of the fundamental group and pointed covers. The correspondence is better than a one to one set mapping in that it will preserve a partial ordering on the two sets. Recall an ordering ($\leq$) a set S is called a partial ordering if it is transitive ($x \leq y$ and $y \leq z$ then $x \leq z$) and each element is equal to itself ($x \leq x$). Any set of subsets of a fixed set inherits the partial order given by the notion of subset. Give this partial ordering to the set of all subgroups of $\pi_1(X, x)$. We also have the following...

Definition 6 Let $[Y, y]$ and $[Z, x]$ be a pair of pointed covers of (X, x). Say that $[Y, y] \preceq [Z, z]$ if and only if there is a covering map $\rho^y_z : (Y, y) \to (Z, z)$.

Notice this explains the seemly wacky use of the the $\preceq$ sign in $Deck(Y \preceq X)$. Notice the the covering space is viewed as "smaller" in this ordering! We may now articulate the...

Theorem 7 Galois Correspondence Suppose X and Y are spaces with Y a normal cover of X. Then the above $\preceq$ ordering forms a partial order on the the pointed covers of (X, x) and there is a one to one partial order preserving correspondence between the subgroups of $Deck[Y \preceq X]$ and the pointed covers $[Z, z]$ satisfying of $[Y, y] \preceq [Z, z] \preceq [X, x]$. Denote the pointed cover corresponding to H as $[X_H, x_h]$, the covering map from (X_H, x_h) to (X, x) as ρ^h_x, and the covering map from (Y, y) to (X_H, x_h) as ρ^y_h. With this notation the following relationships hold between the topology and the algebra.

1. $Deck[Y \preceq X_H] \cong H$
2. $(\rho^y_h)^{-1}(x_h) \equiv H$
3. $X_H \simeq H \backslash Y$
4. $Deck[X_H \preceq X] \cong H \backslash N(H)$
5. $(\rho^h_x)^{-1}(x) \equiv H \backslash \pi_1(Y, y)$

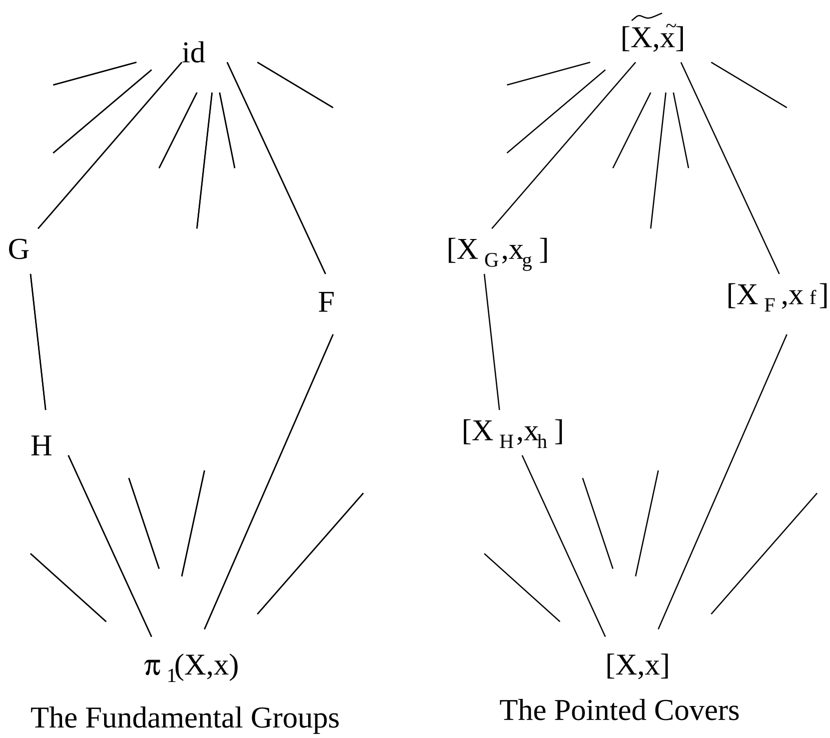

Figure 1: The Galois Correspondence

6. $[X_H, x_h] \equiv H \backslash N(H)$

I should say now that we will make explicit all of these isomorphisms, homeomorphisms, and set bijections (at least from a certain point of view).

Problem 8 Interpret what this correspondence says about normal subgroups and in particular show that X_N is a normal cover of X if and only if N is normal subgroup of $Deck[Y \preceq X]$.

Problem 9 Show if you replace the spaces with fields and the deck groups with the field automorphism groups that this is the Galois correspondence found in field theory.

The interaction with fundamental group is provided via the following theorem.

Theorem 10 Given a space X there exist a simply connected normal cover $\tilde{X}$ with its deck group isomorphic to $\pi_1(X, x)$.

Corollary 11 In particular (from the Deck theorem) for every subgroup H of $\pi_1(X, x)$ we have a space $X_H \simeq \tilde{X}/H$ with $(\rho_x^h)_\star(\pi_1(X_H, x_h)) = H$. Furthermore every cover of X is also covered by $\tilde{X}$, hence $\tilde{X}$ is called X's universal cover. Form this and the Galois correspondence we see that the covering space theory of X is completely determined by $\pi_1(X, x)$.

Problem 12 Find two analogs of this result and its corollary in field theory (Hint: the existence of algebraic closures and the existence of splitting fields).

Problem 13 From the above corollary we know each of the pointed covers described on page 57 of Hatcher corresponds to a cover of the punctured torus as well. Attempt to produce all the covers of the punctured torus corresponding to the covers of the wedge of two circles on page 57, and try to picture them as embedded in three dimensional Euclidean space.

Problem 14 Let G be anyone of $\frac{\mathbf{Z}}{2\mathbf{Z}}$, $\frac{\mathbf{Z}}{3\mathbf{Z}}$, $\frac{\mathbf{Z}}{6\mathbf{Z}}$, or S_3 (where S_3 is the symmetric group an three objects). Draw a diagram representing the partial order on the subgroups G. Form a topological space whose fundamental group is G, and describe all its covers and the explicitly describe the deck groups associated to each cover. Attempt to draw pictures of the spaces involved (these pictures are likely to be in the form of a disjoint union of simple to describe spaces with identifications labeled).