Math 74 Final

1 Hyperbolic Geometry

1. Prove that a set consisting of two distinct horocirlces and a geodesic which are all pairwise tangent is unique up to isometry and compute a finite length which arises naturally in this configuration.

2. Suppose you are given two geodesics γ_1 and γ_2 which do not intersect and are not tangent. Prove that there is a unique geodesic λ intersecting γ_1 at x and γ_2 at y with $d(x, y) = d(\gamma_1, \gamma_2)$. Furthermore prove that λ is the unique geodesic simultaneously perpendicular to γ_1 and γ_2 (think banana!).

3. Let Γ be a subgroup of $Isom(H^2)$ which acts in a deck like way on H^2.

   (a) Prove Γ must have no elliptic elements or reflections.

   (b) Prove that if two non-identity elements of Γ commute then they have the same fixed points.

   (c) Prove that if Γ is commutative then it is conjugate to a group generated by either $a^2 z$, $-a^2\bar{z}$ or $z + a$ (viewed in the upper-half plane model) where a is a real non-zero number. Geometrically describe a commutative Γ's action on H^2.

   (d) Prove that any commutative Γ must have an orbit space homeomorphic to the infinite cylinder or the infinite Mobius strip.

2 Geometrization: Uniqueness

In class we saw that every compact connect surface could be made an G-surface. The goal of the following problem is to prove that the G associated to a given surface is unique. Throughout this problem all compact connected surfaces other than the sphere (S^2), projective plane (P^2), torus (T^2) and Klein bottle (K) will be denoted as M.

1. (a) Prove that any E^2 or H^2-surface must have a non-compact universal cover.

   (b) Prove that S^2 and P^2 cannot be made into an E^2 or H^2-surface.

2. (a) Prove from scratch (using no geometry just topology) that the universal cover of T^2, K or M is not compact (Hint: use the Galois correspondence).
   (b) Prove that T^2, K or M can not be made into an S^2-surface.
3. (a) For any subgroup Γ of $Isom(E^2)$ which acts on E^2 in a deck-like way prove that $\{g^2 \mid g \in \Gamma\}$ is a commutative subset $Isom(E^2)$ (a commutative subset is one where all its elements commute with each other).
   (b) Use this observation to prove that M cannot be made into an E^2-surface (Hint: once again think Galois correspondence).
4. (a) Prove that T^2 covers K.
   (b) Prove that if K is a G-surface then T^2 can be made into a G-surface as well.
   (c) Use exercise 3 of the first section in order to prove that T^2 and K can not be made into H^2-surfaces.

3 Geometrization: Non-compact Existence

Here I'd like to touch on the ideas needed in order to tackle the building of non-compact hyperbolic surfaces which are the orbit space of a group of isometries acting in a "deck like" way; and in particular develop the notion of an earthquake.

1. Prove that there is a unique circle tangent to all three of the geodesics forming an ideal triangle, and find this circle's radius.
2. Prove that two ideal triangles can be geometrically glued together in an infinite number of ways, describe how the circle in the above exercise 1 can be used to quantify the possible glueings, and describe how this situation differs from the situation for finite triangles. (By geometrically glued I mean the indentification space formed via a mapping sending one edge of a given ideal triangle to an edge of a second ideal triangle via an isometry. In otherwords an isometry from the first edge to the second, which simply amounts to an isometry of one geodesic onto another.)
3. (Hard (but I think extremely fun) extra credit.) Look at the pattern we developed for unrolling the punctured torus into the hyperbolic plane.

Using the above parameter, describe how to glue up a collection of ideal triangles in order to form this punctured torus. Now attempt to describe what happens as we change the gluing parameters involved! (Hint: we will always get a topological punctured H^2-torus however we may fail to "fill up" the hyperbolic plane. Explain why.)

4. (Also extra credit) Describe why it is reasonable to call such parameter changes **earthquakes**.