The First Exam

1. **Pick** one of the following groups and call it G: either the dihedral group of order eight

$$D_8 \cong < a, b | a^2, b^4, abab >$$

or the following interesting group of order 16

$$Q_{16} \cong < a, b | a^2, b^8, bab^5 a > .$$

In both these groups every element is in the form $b^m a^n$, and on our web site I have sketched the diagrams of the partially ordered subgroups these groups (where $< c, .., d >$ means the subgroup of G generate by the elements in the set $\{c, ..., d\}$). You should try Q_{16} (especially if you like group theory). Notice Q_{16} contains contains both $D_8 \cong < a, b^2 >$ and the quarternions $\cong < b^2, ab^3 >$ as subgroups, so is bound to be a good time!

   (a) Explicitly construct a space (X_G, x_g) with fundamental group G and use Van Kampen's theorem to prove this space has the correct fundamental group.

   (b) Explicitly construct all of X_G's pointed covers, labeling each vertex with an appropriate $H < a > \in H \backslash \pi_1(X_g, x_g)$ element.

   (c) **As groups**, describe the deck groups over X_G for each of these covers.

   (d) In the universal cover describe the right action of **a** $\pi_1(X, x)$ element on $(\rho_X^{id})^{-1}(x)$ which **fails** to extend to a deck transformation, and describe the deck transformation corresponding to this same element of $\pi_1(X, x)$.

2. Let (Y, y) be a cover of (X, x) in the standard way.

   (a) Find and justify a presentation of $\rho_\star(\pi_1(Y, y))$.

   (b) Describe the conjugacy classes of $\rho_\star(\pi_1(Y, y))$ in $\pi_1(X, x)$.

   (c) We know there is a cover of the punctured torus corresponding to $\rho_\star(\pi_1(Y, y))$, construct it.

   (d) We know there is a cover of the punctured torus corresponding to the cover of (X, x) described in part (13) of the figure on page 57 in Hatcher, construct it.

(e) Construct a cover (Z, z) of (X, x) such that $\rho_\star(\pi_1(Z, z))$ is the commutator subgroup of $< a, b >$ (and justify this assertion).

(f) Prove that (X, x) has uncountablely many distinct covers (Hint: find a subset of the covers of (X, x) which are in one to one correspondence with the set of sequences in the form $\{x_i\}_{i=1}^{\infty}$ with $x_i \in \{0, 1\}$.)