Conformal geometry

It will be useful to observe that our two-dimensional geometries are analytically one-dimensional, though one complex dimension. Recall from multivariate calculus, if we are given a differentiable mapping $\vec{f}(x,y) = (u(x,y), v(x,y))$ between open subsets R^2 its effect on vectors living at the point (x,y) is given by the f's associated Jacobian mapping

$$f_\star = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}.$$

Recall thats this mappings determinant, the Jacobian $Jac(f)$, is the is the distortion factor which arises in an integral when changing coordinates, and further more recall that the Jacobian's sign determines whether $\vec{f}$ changes orientation or not.

We will call $\vec{f}$ *conformal* it preserves angles i.e. if at each point (x,y) in $\vec{f}$'s domain and each pair of vectors v and w at (x,y) we have that

$$\frac{< v, w >}{||v||||w||} = \frac{< f_\star(v), f_\star(w) >}{||f_\star(v)||||f_\star(w)||}.$$

with $< -, - >$ the Euclidean inner-product and $||v|| = \sqrt{< v, v >}$.

Exercise 1 Prove this is equivalent to $f_\star$ being in the the form cA with $A \in O(2)$ or rather at each point $f_\star$ is either in the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \texttt{or} \begin{bmatrix} -a & b \\ b & a \end{bmatrix}.$$

Another way of saying this is that either $u_x = v_y$ and $u_y = -v_x$in which case $\vec{f}$ is called holomorphic or that $u_x = -v_y$ and $u_y = v_x$ in which case $\vec{f}$ is called anti-holomorphic. Notice that the sign of the $Jac(f)$ is positive for a holomorphic mapping and negative for an anti-holomorphic mapping. Hence the holomorphic verse ant-holomorphic distinction is in part one of orientation. When the domain is connected the fact that the Jacobian cannot be zero together with its continuity implies that a conformal mapping is either holomorphic or anti-holomorphic throughout the domain.

The beautiful thing about such functions is that they behave like functions of one variable. In oder to articulate this let we will view the real plane as the complex line with the notation $z = x + iy$ and viewing $\vec{f}$ as $f(z) = u(x,y) + iv(x,y)$. Recall the notations $\bar{z} = x - iy$, $|z| = \sqrt{(z\bar{z})}$, $Im(z) = y$,and $Re(z) = x$. Notice the that our above matrices show up quite naturally...

Exercise 2 Let $\Psi : C \to M_2(R)$ be given by

$$\Psi(x+iy) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

Prove Ψ is an algebra isomorphism (i.e. $\Psi(z+w) = \Psi(z) + \Psi(w)$ and $\Psi(zw) = \Psi(z)\Psi(w)$).

We now define a pair of derivative that interact elegantly with the complex multiplication

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The utility of these derivatives stems in part stems from there interaction with the Jacobian mapping via...

Exercise 3 Let $[a, b]$ denote a vector at the point (x, y). If f is holomorphic show

$$f_\star([a, b]) = \left(\frac{\partial f}{\partial z}\right)(a + ib)$$

and $Jac(f) = |\frac{\partial f}{\partial z}|^2$, where we are evaluating $f_\star$ and$\frac{\partial f}{\partial z}$ at $x + iy$. If f is anti-holomorphic show

$$f_\star([a, b]) = \left(\frac{\partial f}{\partial \bar{z}}\right)(a - ib)$$

and $Jac(f) = -|\frac{\partial f}{\partial \bar{z}}|^2$.

Notice this tells us that holomorphic function are precisely the complex valued complex functions whose derivatives are complex valued as well. It's worth pointing that a function is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}} = 0$ and anti-holomorphic if and only if $\frac{\partial f}{\partial z} = 0$. The heart of the utility of this languages stems form the following...

The Calculus Principle: Holomorphic functions behave as one would expect when view as one dimensional complex functions. In other words if f and g are holomorphic then there associated sums, products, compositions and $\frac{\partial}{\partial z}$ derivatives are also holomorphic. Among holomorphic functions the $\frac{\partial}{\partial z}$ behaves like the usual derivative, i.e. satisfies the chain, product, power, and linearity rules, while the usual 2 dimensional chain rule for a path becomes becomes

$$\frac{dlf(\gamma(t))}{dt} = \frac{\partial f}{\partial z}(\gamma(t))\frac{d\gamma(t)}{dt}.$$

Furthermore real valued functions which are well defined when we replace the real variable with z are holomorphic functions, and are anti-holomorphic if we replace the real variable with $\bar{z}$.

The Riemann Sphere: Since infinity will not play a huge role in our eventual hyperbolic applications I will only mention the following: **More often than not complex function do not live in the complex plane**. For us it will be best to think of our functions as function to and from the one point compactification of the complex plane, denoted $\hat{C}$. $\hat{C}$ is a topological sphere, but we need to understand its conformal structure. Fortunately the stereographic projection map provides a conformal homeomorphism between the sphere minus a point and the plane. It fact its quite a bit better than conformal in that it sends circles on the sphere to circles and lines in the plane and conversely. Hence as a conformal object we are free to view $\hat{C}$ as the usual sphere (called the Riemann sphere when we view it as a conformal object). By a "circle" in $\hat{C}$ we will mean a circle on the sphere or a circle or line in the plane. Many natural conformal mappings are best viewed as conformal mappings of a domain in $\hat{C}$ to another domain in $\hat{C}$, and one should think of the function's representation as $f(z)$ being as being a view of this function in a "coordinate chart".

For us there is a very special group of holomorphic functions that most certainly should be viewed in $\hat{C}$ given by

$$M = \left\{ \frac{az+b}{cz+d} \mid a,b,c,d \in C; ad-bc \neq 0 \right\}.$$

We view this as group under function composition and it is easily verified to be such. Notice that taking care of ∞ is particular easy here since given $f(z) = \frac{az+b}{cz+d}$ we will are forced to define $f(\infty) = a/c$ (when $c \neq 0$ and ∞ otherwise), and we are forced to let $f(-d/c) = \infty$ when $c \neq 0$.

This group is related to a familiar matrix group, and this relationship will prove important in several ways. Let $GL(2,C)$ be the group of two by two invertible matrices with complex entries. Let notice that $\{cI \mid c \in C-0\}$ is a normal subgroup and let $PSL(2,R)$ be the group $GL(2,C)/\{cI \mid c \in C-0\}$, this group is call the projective special linear group.

Exercise 4 Show that the map sending $\frac{az+b}{cz+d}$ to the equivalence class containing

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in $PSL(2, R)$ is an group isomorphism.

It will be useful to gather an arsenal of some basic properties of this group.

Lemma 5 The Oreintation Preserving Mobius Group Lemma

1. $f \in M$ then f is an orientation preserving conformal homeomorphism of $\hat{C}$.

2. If $f \in M$ and $f \neq id$ then f has either one or two fixed points.

3. $f \in M$ is determined by what it does to any three distinct points.

4. Given distinct points $\{z_1, z_2, z_3\}$ and distinct points $\{w_1, w_2, w_3\}$ there exist an $f \in M$ such that $f(z_i) = w_i$.