Math 73, Winter 2004, Midterm Exam

due Tuesday, February 10, 1 PM (in class)

Instructions: You may use the text-book, your notes, or the library, but the written form that you hand in should be well articulated and coherent, and should reflect your understanding of the assigned problems. **The only** person you may discuss the exam with is your instructor. A violation of this will be treated as a violation of the Honor Principle.

There are three sections. For perfect score you should do two problems from each section. (You may do more for extra credit, but clearly indicate the two main problems that you chose.) The graduate students must do the last problem from each section.

Write on one side of your paper, each problem on a separate sheet, and make sure that your name is on every page.

1. Differentiation

Exercise 1.1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(\mathbf{0}) = 0$ (where $\mathbf{0} = (0,0)$) and

$$f(x,y) = \frac{x^2 y}{x^2 + y^2}, \text{ when } (x,y) \neq \mathbf{0}.$$

Show that all the directional derivatives $f'(\mathbf{0}; \mathbf{u})$ exist. Is f differentiable at $\mathbf{0}$?

Exercise 1.2. Suppose that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the conditions of the Inverse Function Theorem. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ denote the inverse function and set

$$J(\mathbf{x}) = \det(Df(\mathbf{x})).$$

Show that

$$\frac{\partial F_1}{\partial y_1}(\mathbf{y}) = \frac{1}{J(\mathbf{x})}\frac{\partial f_2}{\partial x_2}(\mathbf{x}), \quad \frac{\partial F_2}{\partial y_1} = -\frac{1}{J}\frac{\partial f_2}{\partial x_1}, \quad \frac{\partial F_1}{\partial y_2} = -\frac{1}{J}\frac{\partial f_1}{\partial x_2}, \quad \frac{\partial F_2}{\partial y_2} = \frac{1}{J}\frac{\partial f_1}{\partial x_1},$$

where $\mathbf{y} = f(\mathbf{x})$, $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$.

Exercise 1.3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be differentiable. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by the equation

$$F(x,y) = f\left(x, y, g(x,y)\right).$$

(a) Find DF in terms of the partials of f and g.
(b) If $F(x,y) = 0$ for all (x,y), find $D_1 g$ and $D_2 g$ in terms of the partials of f.
(c) Test your answers to (a) and (b) in the case when

$$f(x,y,z) = x^2 + y^2 + z - 1, \quad g(x,y) = 1 - x^2 - y^2.$$

(d) Assume now that A is open in $\mathbb{R}^k$, B is open in $\mathbb{R}^n$, and $f : A \times B \to \mathbb{R}^n$ is differentiable. Suppose also that there exists a differentiable function $g : A \to \mathbb{R}^n$ such that:

$$f(\mathbf{x}, g(\mathbf{x})) = \mathbf{0}, \text{ for all } \mathbf{x} \in A.$$

Find an equation that $Dg(\mathbf{x})$ satisfies. Fully support your answer.

2. Integration in $\mathbb{R}^n$

Exercise 2.1. 1, page 103.

Exercise 2.2. 4, page 132. (This was assigned in Homework #5.)

Exercise 2.3. Let $f : [0,1] \times [0,1] \to \mathbb{R}^2$ be a continuous mapping. Show that the **graph** G of f has measure 0 in $[0,1] \times [0,1] \times \mathbb{R}^2$. Here

$$G = \{\ (\mathbf{x}, \mathbf{y}) \bigm| \mathbf{x} \in [0,1] \times [0,1], \mathbf{y} \in \mathbb{R}^2 \text{ such that } f(\mathbf{x}) = \mathbf{y}\ \}.$$

Exercise 2.4. 7, page 111.

3. Miscellaneous

Exercise 3.1. Show that if $F : \mathbb{R}^n \to \mathbb{R}^m$ is a C^1 function and if the differential $DF(\mathbf{x}) = 0$ for every $\mathbf{x} \in \mathbb{R}^n$, then F is constant.

Exercise 3.2. 3(c), page 103.

Exercise 3.3. Show that the function $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ which is given by $f(X) = X^t X$ is differentiable and find its differential. (Here $M_n(\mathbb{R})$ is the space of $n \times n$ matrices and X^t denotes the transpose of the matrix X.) Use the definition of differential and, if you like, the relationship between the derivative and directional derivatives, but do not use any theorems concerning the existence of the differential.

Hint for Exercise 3.3. There are two ways you can approach this problem. Both rely on the following observation. *View $M_n(\mathbb{R})$ as a vector space, with respect with usual componentwise operations. Then any $A \in M_n(\mathbb{R})$ generates four linear transformation of $M_n(\mathbb{R})$ into itself:*

$$(\star) \qquad B \mapsto A \cdot B, \quad B \mapsto A \cdot B^t, \quad B \mapsto B \cdot A, \text{ and } \quad B \mapsto B^t \cdot A,$$

for $B \in M_n(\mathbb{R})$, and where the dot denotes ordinary matrix multiplication. Then:

Method I. Compute the directional derivatives of f, and hope that you are in the easy case and that the differential is of the type $(\star)$.

Method II. Identify $M_n(\mathbb{R})$ with R^{n^2}, and view f as a map from R^{n^2} to itself; compute the differential of f in these "coordinates", and then identify it using $(\star)$ with appropriate matrix multiplications.

In both cases you have to use the definition on page 43 of differentiability to check that what you were doing is legit.