Handout #6. Tensors and Alternating Tensors

6. Tensors; Tensor product; Alternating Tensors

Let V be an n-dimensional vector space. Recall that a **k-tensor on V** is a multilinear map

$$f : \underbrace{V \times V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}.$$

In particular, a 1-tensor is a linear functional, and a 2-tensor is a bilinear form.

Exercise 6.1. Show that the space $\mathcal{L}^k(V)$ of k-tensors on V is a vector space.

Exercise 6.2. (a) Check that any k-tensor is perfectly determined by its values on all k-tuples of basis vectors $\{\, (e_{i_1}, e_{i_2}, \ldots, e_{i_k}) \,|\, 1 \leq i_1, \ldots i_k \leq n \,\}$.

(b) Let V^* be the dual of V. For $\alpha_1, \alpha_2, \ldots, \alpha_k \in V^*$, define

$$\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_k : V^k \to \mathbb{R}, \quad \alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_k(v_1, v_2, \ldots, v_k) = \prod_{i=1}^{k} \alpha_i(v_i).$$

Show that $\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_k$ is a tensor.

(c) Find the dimension of $\mathcal{L}^k(V)$.

Exercise 6.3. A k-tensor f is said to be **alternating** if

$$f(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -f(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k), \qquad \text{for all } v_1, \ldots, v_k \in V.$$

Assume that $V = \mathbb{R}^3$. Find a basis for the alternating 2- and 3-tensors on $\mathbb{R}^3$.

Exercise 6.4. Let V be an n-dimensional vector space, with basis $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

(a) Show that

$$\det : V^n \to \mathbb{R}, \quad \det(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \det[\mathbf{v}_1, \ldots, \mathbf{v}_n],$$

where the matrix has columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$, is an alternating n-tensor.

(b) Show that every alternating n-tensor f is uniquely determined by the value $f(\mathbf{e}_1, \ldots, \mathbf{e}_n)$.

(c) Conclude that the space of n tensors is one dimensional with the determinant as generator.

Exercise 6.5. Find the dimension of $\mathcal{A}^k(V)$.