Handout #5. Manifolds (with or without boundary)

5. Manifolds in $\mathbb{R}^n$

Recall that a point $\mathbf{p}$ of a k-manifold M in $\mathbb{R}^n$ is called an **boundary point** if there is a coordinate patch $\alpha : U \to V$ on M about $\mathbf{p}$, with U open in $\mathbb{H}^k$ but not in $\mathbb{R}^k$. The following result identifies the boundary points:

Exercise 5.1. Let M be a k-manifold in $\mathbb{R}^n$; let $\alpha : U \to V$ be a coordinate patch about $\mathbf{p}$. If U is open in $\mathbb{H}^k$ and $\mathbf{p} = \alpha(\mathbf{x}_0)$, for $\mathbf{x}_0 \in \mathbb{R}^{k-1} \times 0$, then $\mathbf{p}$ is a boundary point of M.

The next two problems describe ways to cover the sphere S^n with coordinate patches.

Exercise 5.2. Let $M = S^2 = \left\{ (x, y, z) \,\middle|\, x^2 + y^2 + z^2 = 1 \right\} \subset \mathbb{R}^3$. Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ be the "North" and "South" poles, respectively. Denote by β_N and β_S the stereographical projections onto the xy-plane from the North and South poles, respectively.

(a) Find the concrete formulas of these maps:

$$\beta_N : S^2 \setminus \{N\} \to \mathbb{R}^2, \quad \text{ and } \beta_S : S^2 \setminus \{S\} \to \mathbb{R}^2.$$

(b) Find the formulas for the coordinate patches given by $\alpha_N = \beta_N^{-1}$ and $\alpha_N = \beta_N^{-1}$.

(c) Find the formulas of the compositions $\beta_N \circ \alpha_S$ and $\beta_S \circ \alpha_N$, their domains, and show directly that they are of class C^∞.

Exercise 5.3. For $n \geq 2$, let $M = S^n = \left\{ \mathbf{x} = (x_1, x_2, \dots, x_{n+1}) \,\middle|\, \sum_{k=1}^{n+1} x_k^2 = 1 \right\} \subset \mathbb{R}^{n+1}$. Denote by U be the unit open disk in $\mathbb{R}^n$, and for each $i = 1, 2, \dots, n+1$ let

$$\alpha_{\pm i} : U \to \mathbb{R}^{n+1}, \alpha_{\pm i}(x_1, x_2, \dots, x_n) = (x_1, \dots, x_{i-1}, \pm\sqrt{1 - \textstyle\sum_{k=1}^n x_k^2}, x_i, \dots, x_n).$$

These are $2(n+1)$ coordinate patches on S^n.

(a) Show that the above coordinate patches cover all the points of S^n.

(b) Find the formulas for the compositions $\alpha_{\pm j}^{-1} \circ \alpha_{\pm i}$, their domains, and show directly that they are of class C^∞.