Math 68. Algebraic Combinatorics. Fall 2013.

Possible topics for student presentations

1. **Domino tilings of Aztec diamonds.** Counting the number of domino tilings of a given shape is a difficult problem in general. However, for the so-called Aztec diamond, the number of domino tilings has a very simple formula.
   - [Aig] Pages 44–50.

2. *(Taken by James)* **Random walks in $\mathbb{Z}^d$.** The probability that a random walk in $\mathbb{Z}^d$ returns to the origin is 1 for $d = 1, 2$, but strictly less than 1 for $d \geq 3$. In other words, you shouldn't get drunk unless you move in at most two dimensions.
   - [Aig] Pages 85–89.

3. *(Taken by Kellie and Amanda)* **Walks in graphs.** The number of walks of given length between two vertices of a graph can be expressed in terms of the eigenvalues of its adjacency matrix. This is a nice connection of combinatorics and linear algebra.
   - [St] Chapter 1.

4. **Cubes and the Radon transform.**
   - [St] Chapter 2.

5. **The Gessel-Viennot method.** This is a remarkable formula to enumerate n-tuples of nonintersecting lattice paths. The answer is given by a determinant of binomial coefficients, and the proof is based on the combinatorics of involutions.
   - [Aig] Section 5.4.
   - [EC1] Section 2.7.
   - I. Gessel and G. Viennot, Binomial determinants, paths, and hook length formulae, *Advances in Math.* 58 (1985), 300–321.

6. **The descent number and the major index.** We say that i is a *descent* of a permutation $\pi \in \mathcal{S}_n$ if $\pi_i > \pi_{i+1}$. The *major index* of π, denoted $\text{maj}(\pi)$, is defined as the sum of all descents in π. For example, $\text{maj}(12\cdots n) = 0$ and $\text{maj}(n\cdots 21) = 1+2+\cdots+(n-1)$. On the other hand, an *inversion* of π is a pair (i, j) such that $i < j$ and $\pi_i > \pi_j$. The *inversion number* of π, denoted $\text{inv}(\pi)$, is the number of inversions of π.

   The goal of this project is to show that for any value k, the number of permutations $\pi \in \mathcal{S}_n$ with $\text{maj}(\pi) = k$ is the same as the number of permutations $\pi \in \mathcal{S}_n$ with $\text{inv}(\pi) = k$. In other words, the major index maj is equidistributed with the number of inversions inv, that is,
$$\sum_{\pi \in \mathcal{S}_n} q^{\text{maj}(\pi)} = \sum_{\pi \in \mathcal{S}_n} q^{\text{inv}(\pi)}.$$

   A stronger version of this is the fact that the joint distribution of maj and inv is symmetric, that is,
$$\sum_{\pi \in \mathcal{S}_n} q^{\text{maj}(\pi)} t^{\text{inv}(\pi)} = \sum_{\pi \in \mathcal{S}_n} q^{\text{inv}(\pi)} t^{\text{maj}(\pi)}.$$

   - [EC1] Proposition 1.4.6.

- D. Foata and M.-P. Schutzenberger, Major index and inversion number of permutations, *Math. Nach.* 83 (1978), 143–159.

7. **Viennot's geometric construction of the RSK correspondence.** In class we will discuss the RSK algorithm, which gives a correspondence between permutations and pairs of standard Young tableaux. A beautiful geometric description of this correspondence is due to Viennot.
   - Section 3.6 of [Bruce Sagan, *The Symmetric group*, Springer, second edition, 2001].
8. *(Taken by Jacob)* **Increasing and decreasing subsequences of permutations.** This theory is an application of the Robinson-Schensted correspondence (or RSK algorithm).
   - [BS] Section 5.
9. *(Taken by Justin and Kate)* **The transfer-matrix method.** An application of counting walks in graphs to other problems in enumerative combinatorics.
   - [EC1] Section 4.7.
10. *(Taken by Omar)* **A combinatorial proof of the Lagrange Inversion Formula.**
    - [EC2] Theorem 5.4.2 (choose second or third proof).
11. *(Taken by Sam)* **A proof of the hook-length formula.**
    - [Greene, Nijenhuis, Wilf, A probabilistic proof of a formula for the number of Young tableaux of a given shape, *Adv. in Math.* 31 (1979) 104–109]
    - You can also check [Novelli, Pak, Stoyanovskii, A direct bijective proof of the hook-length formula, *Disc. Math. Comp Sci.* 1 (1997), 53–67].
12. *(Taken by Sean)* **Connections of RSK with representation theory and Schur functions.**
13. **A combinatorial proof of the unimodality of the q-binomial coefficients.**
    - [D. Zeilberger, Kathy O'Hara's constructive proof of the unimodality of the Gaussian Polynomials, *The American Mathematical Monthly* 96, 590–602].
14. Read a paper from a combinatorics journal and present it in class. You are encouraged to talk with me to pick a suitable paper. Here are some interesting journals that you can find in the library or online.
    - *Journal of Combinatorial Theory A*,
    - *Electronic Journal of Combinatorics*,
    - *Journal of Algebraic Combinatorics*,
    - *European Journal of Combinatorics*,
    - *Annals of Combinatorics*,
    - *Discrete Mathematics.*