Math 68. Algebraic Combinatorics.

Problem Set 2. Due on Thursday, 10/25/2007.

1. Let $F(z) = \sum_{n\geq 0} a_n z^n$ with $a_0 = 0$. Show that $F(z)$ has compositional inverse $F^{<-1>}(z) = \sum_{n\geq 0} b_n z^n$ with $b_0 = 0$ if and only if $a_1 \neq 0$.

2. Find the ordinary generating function of the sequence $a_n = 2 \cdot 3^n - n^2$ (for $n \geq 0$) in a simple, closed form.

3. Consider the recurrence $a_{n+3} = 3a_{n+2} - 4a_n$, with initial conditions $a_0 = 1$, $a_1 = 2$, $a_2 = 6$. Find the ordinary generating function $\sum_{n\geq 0} a_n z^n$ and the expression of the general term a_n.

4. Find a generating function $A(z)$ such that the coefficient of z^{100} is the number of ways to give change of a dollar using cents, nickels, dimes, and quarters.

5. Given two sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$, its Hadamard product is the sequence $\{a_n b_n\}_{n\geq 0}$. Show that if $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ have rational generating functions, then so does their Hadamard product.

6. Find an expression for $S(n, k)$ (the Sterling number of the second kind) by extracting the coefficient of z^n in the exponential generating function for set partitions with k blocks.

7. Prove that
$$\prod_{n\geq 0} \frac{1}{1 - z^n} = \sum_{k\geq 0} \frac{z^{k^2}}{[(1-z)\cdots(1-z)^k]^2}.$$

8. Let h_n be the number of ways to choose a permutation π of $[n]$ and a subset S of $[n]$ such that if $i \in S$, then $\pi(i) \notin S$. Find an expression for the exponential generating function $\sum_{n\geq 0} h_n \frac{z^n}{n!}$.