Exercise 6.6. For a sequence $(a_n)_n$ the number $\liminf a_n$ is the smallest number to which a subsequence of $(a_n)_n$ can converge to, and $\limsup a_n$ is the biggest such number:

$$\liminf a_n = \inf \left\{ l \in \mathbb{R} \cup \{\pm\infty\} \mid \exists (n_k)_k \text{ with } \lim_k a_{n_k} = l \right\},$$

$$\limsup a_n = \sup \left\{ l \in \mathbb{R} \cup \{\pm\infty\} \mid \exists (n_k)_k \text{ with } \lim_k a_{n_k} = l \right\}.$$

(*Hint.* You may find useful to show/know that if $\liminf a_n > a$ then at most finitely many terms of $(a_n)_n$ are smaller than a.)

Remark. In Exercise 6.5 you were asked to show:

> **Cauchy's criterion for convergence.** *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

One of its immediate consequences is that $\mathbb{R}$ is complete.

Exercise 6.7. What other subsets of $\mathbb{R}$ do you think are also complete?

Exercise 6.8. The purpose of this exercise is to make sense of the following object:

$$\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}}$$

It is known as a **continued fraction**, and is to be interpreted as the real number a which is the limit of the sequence:

$$a_1 = \frac{1}{2},\ a_2 = \cfrac{1}{2+\cfrac{1}{2}} = \frac{2}{5},\ a_3 = \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}},\ \dots .$$

(a) Find a *recursion formula*, that is a formula that computes a_{n+1} as a function $f(a_n)$ of the preceding term (with $a_1 = 1/2$ given).
(b) Show that if $a_n > \sqrt{2}-1$ then $a_{n+1} < \sqrt{2}-1$, and vice versa.
(c) Use the recursion formula to compare a_{n+2} with a_n. (*Hint.* You should end up with deciding the sign of $g(x) = x^2 + 2x - 1, x \in \mathbb{R}$.)
(d) Conclude that $(a_{2n})_n$ is increasing and $(a_{2n+1})_n$ is decreasing.
(e) Show that $|a_{n+2} - a_{n+1}| < |a_{n+1} - a_n|/4$, for all $n \geq 1$. Conclude that the sequence is Cauchy and find its limit.