3. Real numbers (continued)

Exercise 3.3. Show that the set $\mathbb{N}$ is **unbounded**. That is, show that there is no natural number N such that $n \leq N$ for all $n \in \mathbb{N}$. Can you bound $\mathbb{N}$ with a *real number* X instead of N?

Exercise 3.4. Use the Exercise 3.3 to prove the **Archimedean property of $\mathbb{R}$**:

If $x, y \in \mathbb{R}$, $x > 0$, then there is a positive integer n such that $y < nx$.

Exercise 3.5. Show that between any two rational numbers $r_1 < r_2$ there is always another rational number. What can you conclude about the representation of rational numbers as points on a line?

Note. As $\sqrt{2}$ shows, after representing the rationals on a line some "gaps" (actually a whole lot of them!) are present. The **irrational numbers**, or the real numbers that are not rational, are the mathematical "fillings" for these gaps.

Exercise 3.6. Define $x^{-1} = 1/x$, for $x > 0$, x real and not rational. (This is a bit long. Articulate well the main ideas and finish all the details when you get home.)

4. Limits

Exercise 4.1. Show using the definition that the sequence $(a_n)_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$ is not convergent.

Exercise 4.2. Show that if a sequence $(a_n)_{n=1}^{\infty}$ converges to L, then any of its subsequences $(a_{n_k})_{n=1}^{\infty}$ converges to the same limit.

Exercise 4.3. If a sequence $(a_n)_{n=1}^{\infty}$ of real numbers converges, then the set $\{a_n \mid n \in \mathbb{N}\}$ is bounded.