15. Limits of Functions

Exercise 15.1. Show that if $f_n \xrightarrow{\text{u}} f$ then $f_n \xrightarrow{\text{ptw}} f$. Is the converse true?

Exercise 15.2. Let K be a compact subset of a metric space. Show that the supremum norm,

$$\|f\|_\infty = \sup_{x \in K} |f(x)|, \text{ for } f \in C(K),$$

is a norm on $C(K)$.

Exercise 15.3. Let K be a compact subset of a metric space. Show that (f_n) in $C(K)$ converges uniformly to f if and only if $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$.

Exercise 15.4. Assume that $f_n \xrightarrow{\text{u}} f$, with all the f_n's continuous. Show that f is continuous.

Exercise 15.5. Let (f_n) be a sequence of continuous functions $f : S \to \mathbb{R}^m$. If $\sum_{n=1}^{\infty} f_n$ converges uniformly, then it is continuous.