Exercise 14.9. Show that the following function is Riemann integrable:

$$f : [0,1] \to \mathbb{R}, \quad f(x) = \begin{cases} 1 & , \quad \text{if } x = \frac{1}{2}, \\ 0 & , \quad \text{elsewhere.} \end{cases}$$

Exercise 14.10. Show that the following function is Riemann integrable:

$$f : [0,1] \to \mathbb{R}, \quad f(x) = \begin{cases} 1 & , \quad \text{if } x = \frac{1}{n}, \text{ with } n \in \mathbb{N}, \\ 0 & , \quad \text{elsewhere.} \end{cases}$$

Exercise 14.11. Show that the following function is Riemann integrable:

$$f : [0,1] \to \mathbb{R}, \quad f(x) = \begin{cases} 0 & , \quad \text{if } x \notin \mathbb{Q}, \\ \dfrac{1}{q} & , \quad \text{if } x = \dfrac{p}{q} \in \mathbb{Q}, \text{ in the lowest terms.} \end{cases}$$

Exercise 14.12. The goal here is to show the following part of Lebesgue's theorem:

If the bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable then its set of points of discontinuity has measure zero.

(a) For each $k \in N$ show that there are a partition P_k and step functions $l_k(x) \leq f(x) \leq u_k(x)$, for all $x \in [a,b]$, such that $\int_a^b \big(u_k(x) - l_k(x)\big)dx < 4^{-k}$.

(b) What estimation can you get for the length (measure) of the sets

$$B_k = \big\{x \,\big|\, \big(u_k(x) - l_k(x)\big) > 2^{-k}\big\}?$$

(c) Show that $B = \bigcap_{k \in \mathbb{N}} \text{interior}(B_k)$ has measure zero.

(d) Can you conclude that B coincides with the set of points of discontinuity of f?

Exercise 14.13. Prove that f is continuous at x if and only if $\text{osc}(f, x) = 0$.