Group Work Tip #2. Encourage everybody in the group to participate in the discussion and listen to what the others are saying.

Exercise 13.8. Prove:

> **Another Mean Value Theorem.** *If f and g are continuous functions on $[a,b]$ which are differentiable on (a,b), then there is a point $c \in (a,b)$ such that*
> $$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} \quad \text{(assume } g(a) \neq g(b)\text{)}.$$

(*Hint.* Consider $h(x) = (f(b)-f(a))\,g(x) - (g(b)-g(a))\,f(x)$.)

14. Riemann integral

Exercise 14.1. Consider the function $f : [0,1] \to [0,1]$, $f(x) = x$. Consider also the equidistant partition of $[0,1]$

$$P = \{\, x_0 = 0, x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \ldots, x_{n-1} = \frac{n-1}{n}, x_n = 1 \,\}.$$

Compute the **lower and upper sums** $L(f,P)$ and $U(f,p)$, respectively. How do these values compare with $1/2$?

Exercise 14.2. Show that we always have:

$$L(f,P) \leq I(f,P,X) \leq U(f,P).$$

Exercise 14.3. A partition R is a **refinement** of a partition P if $P \subset R$. Show that in this case:

$$L(f,P) \leq L(f,R) \leq U(f,R) \leq U(f,P).$$