Exercise 12.10. Prove:

> **The Intermediate Value Theorem.** If $f : [a, b] \to \mathbb{R}$ is a continuous function with $f(a) < 0 < f(b)$, then there exists a point $c \in (a, b)$ such that $f(c) = 0$.

(*Hint*. Consider the set $A = \{x \in [a, b] \mid f(x) < 0\}$.)

Exercise 12.11. One morning, exactly at sun rise, a monk started from the base of a tall mountain to climb towards a monastery on the top of the mountain. He climbed the narrow spiral path around the mountain at various speeds, with stops, and exactly at sun set he reached the top. After a number of days of prayer, exactly at sun rise, he started his way down. He descended at various speeds, with stops, and exactly at sun set he reached the base of the mountain. Assume that the days have the same length. Show that along the path there is a point that the monk reaches at the same time of the day, both during the ascent and during the descent.

Exercise 12.12. Let A be a subset of $\mathbb{R}$. The **characteristic function of** A is the function

$$\chi_A(x) = \begin{cases} 1 & , \quad \text{if } x \in A \\ 0 & , \quad \text{if } x \notin A \end{cases} .$$

Find the discontinuities of $\chi_{[0,1]}$ and $\chi_{\mathbb{Q}}$.

Exercise 12.13. Find the discontinuities of $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & , \quad \text{if } x \notin \mathbb{Q}, \\ \dfrac{1}{q} & , \quad \text{if } x = \dfrac{p}{q} \in \mathbb{Q}, \text{ in the lowest terms.} \end{cases}$$

(*Hint*. Consider a sequence $(r_n)_n$ of rational numbers, where $r_n = p_n/q_n$ in the lowest terms, with $q_n \in \mathbb{N}$, which converges to an irrational number x. What can you say about the sequence $(q_n)_n$?)