12. Continuous functions

Exercise 12.1. Let $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Consider the linear transformation associated to A:

$$T_A : \mathbb{R}^2 \to \mathbb{R}^2, T_A(\mathbf{x}) = A \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \text{ where } \mathbf{x} = (x_1, x_2).$$

(a) Show that there is a constant C such that

$$\| T_A(\mathbf{x}) - T_A(\mathbf{y}) \| \leq C \| \mathbf{x} - \mathbf{y} \|.$$

Conclude that T_A is continuous.

(b) Repeat (a) above but now with an arbitrary matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Exercise 12.2. Review linear transformations from Section 1.4. Prove that any linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is continuous.