Exercise 10.4. Consider the interval $I = [0, 1]$. Perform the following inductive construction: at *Step 1* remove from I the interval $U_1 = (\frac{1}{3}, \frac{2}{3})$; at *Step 2* remove from the set obtained in Step 1 the union $U_2 = (\frac{1}{9}, \frac{2}{9}) \bigcup (\frac{7}{9}, \frac{8}{9})$; and so on, at *Step n* removing the middle third open interval out of each of the 2^{n-1} closed intervals that were obtained in the previous step; call the union of the removed open intervals U_n. Define the **Cantor set** as

$$C = [0, 1] \setminus \bigcup_{n=1}^{\infty} U_n.$$

(a) Show that C is compact.
(b) Show that C consists only of cluster points. (Closed sets with this property are called **perfect**.)
(c) What is the total length of all the intervals removed in the infinite process of the construction of C?
(d) What is the interior of C?
(e) Show that we can describe C as all real numbers in $[0, 1]$ whose **ternary expansion** (that is 'decimal' expansion with 'decimals' taking value in the set $\{0, 1, 2\}$) consists only of 0's and 2's. In other words, show that C is the set of all real numbers x that can be written as

$$x = 0.x_1x_2x_3 \ldots x_n \ldots = x_1\frac{1}{3} + x_2\frac{1}{3^2} + \cdots + x_n\frac{1}{3^n} + \cdots, \text{ with } x_n \in \{0, 2\}.$$

(f) Show that $|C| = |[0, 1]|$. Indeed, write first the elements of $[0, 1]$ in **binary expansion**,

$$y = 0.y_1y_2y_3 \ldots y_n \ldots = y_1\frac{1}{2} + y_2\frac{1}{y^2} + \cdots + y_n\frac{1}{2^n} + \cdots, \text{ with } y_n \in \{0, 1\}.$$

Next construct a map $[0, 1] \to C$. (*Note.* You can use Schroeder-Bernstein theorem if you want.)

Exercise 10.5. (Fun Problem) Is there an uncountable closed subset of $[0, 1]$ which contains no rational number?

11. The number e

Exercise 11.1. Show the following estimation of the rate of convergence of the partial sums that appear in the series definition of e:

$$0 < e - \left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!}\right) < \frac{1}{n! \cdot n}. \tag{2}$$

Exercise 11.2. Show that e is irrational.

(*Hint.* Use the series definition of e. Assume by contradiction that $e = p/q$, in the lowest terms. Multiply (2) of the previous exercise with $q!$, to get a contradiction.)

Note. The number e is actually **transcendental**, which means that e is not the root of any polynomial with rational coefficients, but to prove it is a lot harder. A root of a polynomial with rational coefficients is called an **algebraic number**. Show that the set of algebraic numbers is countable, by proving first that the set of polynomials with rational coefficients is countable. Conclude that the transcendental numbers are uncountable.

Exercise 11.3. Assume the following: if $(a_n)_n$ is a sequence of strictly positive real numbers, and if $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \lim_{n\to\infty} \sqrt[n]{a_1 a_2 \ldots a_n} = L.$$

Show that

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}} = e.$$