10. Cardinality

Exercise 10.1. Show that $|\mathbb{N}| = |\mathbb{N}^*|$ and $|\mathbb{Z}| = |\mathbb{N}^*|$. (In both cases construct an explicit bijective function between the two sets under discussion.)

Exercise 10.2. Show that a countable union of countable sets is countable.

Exercise 10.3. Show that $|\mathbb{Q}| = |\mathbb{N}^*|$.

Exercise 10.4. Consider the interval $I = [0, 1]$. Perform the following inductive construction: at *Step 1* remove from I the interval $U_1 = (\frac{1}{3}, \frac{2}{3})$; at *Step 2* remove from the set obtained in Step 1 the union $U_2 = (\frac{1}{9}, \frac{2}{9}) \bigcup (\frac{7}{9}, \frac{8}{9})$; and so on, at *Step n* removing the middle third open interval out of each of the 2^{n-1} closed intervals that were obtained in the previous step; call the union of the removed open intervals U_n. Define the **Cantor set** as

$$C = [0, 1] \setminus \bigcup_{n=1}^{\infty} U_n.$$

(a) Show that C is compact.
(b) Show that C consists only of cluster points. (Closed sets with this property are called **perfect**.)
(c) What is the total length of all the intervals removed in the infinite process of the construction of C?
(d) What is the interior of C?
(e) Show that we can describe C as all real numbers in $[0, 1]$ whose **ternary expansion** (that is 'decimal' expansion with 'decimals' taking value in the set $\{0, 1, 2\}$) consists only of 0's and 2's. In other words, show that C is the set of all real numbers x that can be written as

$$x = 0.x_1x_2x_3 \ldots x_n \ldots = x_1\frac{1}{3} + x_2\frac{1}{3^2} + \cdots + x_n\frac{1}{3^n} + \cdots, \text{ with } x_n \in \{0, 2\}.$$

(f) Show that $|C| = |[0, 1]|$. Indeed, write first the elements of $[0, 1]$ in **binary expansion**,

$$y = 0.y_1y_2y_3 \ldots y_n \ldots = y_1\frac{1}{2} + y_2\frac{1}{y^2} + \cdots + y_n\frac{1}{2^n} + \cdots, \text{ with } y_n \in \{0, 1\}.$$

Next construct a map $[0, 1] \to C$. (*Note.* You can use Schroeder-Bernstein theorem if you want.)