Group Work Tip #3. Try the following: the complete solution of a problem is written down by **one and only one** member of the group! The others pay attention or offer corrections. Then all move to the next problem and change the writer!

9. Series

Exercise 9.1. (Test #0 for the convergence of a series.) Show that $\lim_{n\to\infty} a_n = 0$ is necessary for $\sum_{n=1}^{\infty} a_n$ to converge. (Another way of saying the same thing is: if $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.)

Exercise 9.2. Use Cauchy's criterion for convergence of a sequence to get an equivalent statement for the convergence of a series.

Exercise 9.3. (Second part of the **Root test**, the **Test #3** for convergence of a series.) Prove that if $L = \limsup \sqrt[n]{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Exercise 9.4. (Condensation criterion of Cauchy.) Let $(a_n)_{n=1}^{\infty}$ be a monotone decreasing sequence of positive real numbers. Show that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=0}^{\infty} 2^k\, a_{2^k}$ converges.

(*Hint*. Before proving the exercise, verify the correctness of its statement in the case of harmonic series. Next recall how one proves that the harmonic series diverges by combining in a certain particular way its terms. Generalize this trick and obtain the result required by the exercise.)

Exercise 9.5. Can you give examples of convergent series for which any rearrangement has the same sum?