

Variance of Discrete Random Variables

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Variance

Definition

Let X be a numerically valued random variable with expected value $\mu = E(X)$. Then the *variance* of X , denoted by $V(X)$, is

$$V(X) = E((X - \mu)^2) .$$

Standard Deviation

The *standard deviation* of X , denoted by $D(X)$, is $D(X) = \sqrt{V(X)}$. We often write σ for $D(X)$ and σ^2 for $V(X)$.

Calculation of Variance

Theorem. *If X is any random variable with $E(X) = \mu$, then*

$$V(X) = E(X^2) - \mu^2 .$$

Poisson Distribution

- Let X be a Poisson random variable with parameter λ .
- What is $V(X)$?

Properties of Variance

Theorem. *If X is any random variable and c is any constant, then*

$$V(cX) = c^2V(X)$$

and

$$V(X + c) = V(X) .$$

Theorem. *Let X and Y be two independent random variables.
Then*

$$V(X + Y) = V(X) + V(Y).$$

Example

- Let X be an n Bernoulli trials process.
- What is $V(X)$?

Theorem. Let X_1, X_2, \dots, X_n be an independent trials process with $E(X_j) = \mu$ and $V(X_j) = \sigma^2$. Let

$$S_n = X_1 + X_2 + \dots + X_n$$

be the sum, and

$$A_n = \frac{S_n}{n}$$

be the average. Then

$$E(S_n) = n\mu ,$$

$$V(S_n) = n\sigma^2 ,$$

$$E(A_n) = \mu ,$$

$$V(A_n) = \frac{\sigma^2}{n} .$$

Example

- Let T denote the number of trials until the first success in a Bernoulli trials process.
- What is the variance of T ?

Continuous Random Variables

Expected Value

Definition. Let X be a real-valued random variable with density function $f(x)$. The expected value $\mu = E(X)$ is defined by

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x) dx ,$$

provided the integral

$$\int_{-\infty}^{+\infty} |x| f(x) dx$$

is finite.

Properties

- If X and Y are real-valued random variables and c is any constant, then

$$E(X + Y) = E(X) + E(Y) ,$$

$$E(cX) = cE(X) .$$

- More generally, if X_1, X_2, \dots, X_n are n real-valued random variables, and c_1, c_2, \dots, c_n are n constants, then

$$E(c_1X_1 + c_2X_2 + \dots + c_nX_n) = c_1E(X_1) + c_2E(X_2) + \dots + c_nE(X_n) .$$

Example

- Suppose Mr. and Mrs. Lockhorn agree to meet at the Hanover Inn between 5:00 and 6:00 P.M. on Tuesday.
- Suppose each arrives at a time between 5:00 and 6:00 chosen at random with uniform probability.
- Let Z be the random variable which describes the length of time that the first to arrive has to wait for the other.
- What is $E(Z)$?

Expectation of a Function of a Random Variable

Theorem. *If X is a real-valued random variable and if $\phi : \mathbf{R} \rightarrow \mathbf{R}$ is a continuous real-valued function with domain $[a, b]$, then*

$$E(\phi(X)) = \int_{-\infty}^{+\infty} \phi(x) f_X(x) dx ,$$

provided the integral exists.

Expectation of the Product of Two Random Variables

Theorem. *Let X and Y be independent real-valued continuous random variables with finite expected values. Then we have*

$$E(XY) = E(X)E(Y) .$$

Example

- Let $Z = (X, Y)$ be a point chosen at random in the unit square.
- What is $E(X^2Y^2)$?

Variance

Definition. Let X be a real-valued random variable with density function $f(x)$. The variance $\sigma^2 = V(X)$ is defined by

$$\sigma^2 = V(X) = E((X - \mu)^2) .$$

Computation

Theorem. *If X is a real-valued random variable with $E(X) = \mu$, then*

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx .$$

Properties of the variance

- If X is a real-valued random variable defined on Ω and c is any constant, then

$$\begin{aligned}V(cX) &= c^2V(X) , \\V(X + c) &= V(X) .\end{aligned}$$

- If X is a real-valued random variable with $E(X) = \mu$, then

$$V(X) = E(X^2) - \mu^2 .$$

- If X and Y are independent real-valued random variables on Ω , then

$$V(X + Y) = V(X) + V(Y) .$$