Expected Value

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Definition

- Let X be a numerically-valued discrete random variable with sample space Ω and distribution function $m(x)$. The *expected value* $E(X)$ is defined by

$$E(X) = \sum_{x \in \Omega} x m(x) \ ,$$

  provided this sum converges absolutely.

- We often refer to the expected value as the *mean*, and denote $E(X)$ by μ

Example

Suppose that we toss a fair coin until a head first comes up, and let X represent the number of tosses which were made. What is $E(X)$?

Example

Suppose that we flip a coin until a head first appears, and if the number of tosses equals n, then we are paid 2^n dollars. What is the expected value of the payment?

Example

- Let T be the time for the first success in a Bernoulli trials process.

- Assign the geometric distribution

$$m(j) = P(T = j) = q^{j-1}p \ .$$

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- Thus,

$$\begin{aligned} E(T) &= 1 \cdot p + 2qp + 3q^2p + \cdots \\ &= \frac{1}{p} \end{aligned}$$

Expectation of a Function of a Random Variable

- Suppose that X is a discrete random variable with sample space Ω, and $\phi(x)$ is a real-valued function with domain Ω. Then $\phi(X)$ is a real-valued random variable. What is it's expectation?

Example

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X	Y
HHH	1
HHT	2
HTH	3
HTT	2
THH	2
THT	3
TTH	2
TTT	1

Theorem. *If X is a discrete random variable with sample space Ω and distribution function $m(x)$, and if $\phi : \Omega \to \mathbb{R}$ is a function, then*

$$E(\phi(X)) = \sum_{x \in \Omega} \phi(x) m(x) \ ,$$

provided the series converges absolutely.

The Sum of Two Random Variables

We flip a coin and let X have the value 1 if the coin comes up heads and 0 if the coin comes up tails. Then, we roll a die and let Y denote the face that comes up. What does $X + Y$ mean, and what is its distribution?

Theorem. *Let X and Y be random variables with finite expected values. Then*

$$E(X+Y)=E(X)+E(Y)\ ,$$

and if c is any constant, then

$$E(cX)=cE(X)\ .$$

Sketch of the Proof

Suppose that

$$\Omega_X = \{x_1, x_2, \ldots\}$$

and

$$\Omega_Y = \{y_1, y_2, \ldots\} \ .$$

$$\begin{aligned} E(X+Y) &= \sum_j \sum_k (x_j + y_k) P(X = x_j,\ Y = y_k) \\ &= \sum_j \sum_k x_j P(X = x_j,\ Y = y_k) \\ &+ \sum_j \sum_k y_k P(X = x_j,\ Y = y_k) \\ &= \sum_j x_j P(X = x_j) + \sum_k y_k P(Y = y_k) \ . \end{aligned}$$

The Sum of A Finite Number of Random Variables

Theorem. *The expected value of the sum of any finite number of random variables is the sum of the expected values of the individual random variables.*

Example

Let Y be the number of fixed points in a random permutation of the set $\{a, b, c\}$. Find the expected value of Y.

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X			Y
a	b	c	3
a	c	b	1
b	a	c	1
b	c	a	0
c	a	b	0
c	b	a	1

Bernoulli Trials

Theorem. *Let S_n be the number of successes in n Bernoulli trials with probability p for success on each trial. Then the expected number of successes is np. That is,*

$$E(S_n) = np \ .$$

Poisson Distribution

- The expected value of a Poisson distribution with parameter λ equals λ.

Independence

Theorem. *If X and Y are independent random variables, then*

$$E(X \cdot Y) = E(X)E(Y) \ .$$

Sketch of the proof

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$$E(X \cdot Y) \quad = \quad \sum_j \sum_k x_j y_k P(X = x_j) P(Y = y_k)$$

$$
\begin{aligned}
&= \left(\sum_j x_j P(X = x_j)\right) \left(\sum_k y_k P(Y = y_k)\right) \\
&= E(X)E(Y) \ .
\end{aligned}
$$

Example

A coin is tossed twice. $X_i = 1$ if the ith toss is heads and 0 otherwise. What is $E(X_1 \cdot X_2)$?

Example

Consider a single toss of a coin. We define the random variable X to be 1 if heads turns up and 0 if tails turns up, and we set $Y = 1 - X$. What is $E(X \cdot Y)$?

Conditional Expectation

If F is any event and X is a random variable with sample space $\Omega = \{x_1, x_2, \ldots\}$, then *the conditional expectation given* F is defined by

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Theorem. *Let X be a random variable with sample space Ω. If F_1, F_2, . . . , F_r are events such that $F_i \cap F_j = \emptyset$ for $i \neq j$ and $\Omega = \cup_j F_j$, then*

$$E(X) = \sum_j E(X|F_j) P(F_j) \ .$$

Martingales

- Recall that Peter and Paul play heads or tail.

- Let S_1, S_2, . . . , S_n be Peter's accumulated fortune in playing heads or tails. Then

$$E(S_n|S_{n-1} = a, \ldots, S_1 = r) = \frac{1}{2}(a+1) + \frac{1}{2}(a-1) = a \ .$$

- Peter's expected fortune after the next play is equal to his present fortune.

- We say the game is *fair*. A fair game is also called a *martingale*.

Problem

In a version of roulette in Las Vegas, a player bets on red or black. Half of the numbers from 1 to 36 are red, and half are black. If a player bets a dollar on black, and if the ball stops on a black number, he gets his dollar back and another dollar. If the ball stops on a red number or on 0 or 00 he loses his dollar. Find the expected winnings for this bet.

Problem

You have 80 dollars and play the following game. An urn contains two white balls and two black balls. You draw the balls out one at a time without replacement until all the balls are gone. On each draw, you bet half of your present fortune that you will draw a white ball. What is your expected final fortune?