Important Distributions

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Continuous Conditional Probability (cont'd)

Theorem. *(**Marginal Densities**) Let X and Y be jointly continuous random variables, with joint density function $f_{X,Y}$. Then the (marginal) density f_X of X satisfies*

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy,$$

for all $x \in \mathbb{R}$. A similar formula holds for f_Y.

Example

- Choose a point $\omega = (\omega_1, \omega_2)$ at random from the unit square. Set $X_1 = \omega_1^2$, $X_2 = \omega_2^2$, and $X_3 = \omega_1 + \omega_2$.
  - Are X_1 and X_2 independent?
  - Are X_1 and X_3 independent?

Function of Independent Random Variables

Theorem. *Let* $X_1, X_2, \ldots, X_n$ *be mutually independent continuous random variables and let* $\phi_1(x), \phi_2(x), \ldots, \phi_n(x)$ *be continuous functions. Then* $\phi_1(X_1)$, $\phi_2(X_2), \ldots, \phi_n(X_n)$ *are mutually independent.*

Discrete Uniform Distribution

- All outcomes of an experiment are equally likely.
- If X is a random variable which represents the outcome of an experiment of this type, then we say that X is *uniformly distributed*.
- If the sample space S is of size n, where $0 < n < \infty$, then the distribution function $m(\omega)$ is defined to be $1/n$ for all $\omega \in S$.

Binomial Distribution

- The distribution of the random variable which counts the number of heads which occur when a coin is tossed n times, assuming that on any one toss, the probability that a head occurs is p.

- The distribution function is given by the formula

$$b(n, p, k) = \binom{n}{k} p^k q^{n-k} ,$$

where $q = 1 - p$.

Geometric Distribution

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$$\begin{aligned} P(T=1) &= p\ , \\ P(T=2) &= qp\ , \\ P(T=3) &= q^2p\ , \end{aligned}$$

and in general,

$$P(T=n) = q^{n-1}p\ .$$

Example

Suppose a line of customers waits for service at a counter. It is often assumed that, in each small time unit, either 0 or 1 new customers arrive at the counter. The probability that a customer arrives is p and that no customer arrives is $q = 1 - p$. Let T be the time until the next arrival What is the probability that no customer arrives in the next k time units, that is, for $P(T > k)$.

Negative Binomial Distribution

- Suppose we are given a coin which has probability p of coming up heads when it is tossed.
- We fix a positive integer k, and toss the coin until the kth head appears.
- Let X represent the number of tosses. When $k = 1$, X is geometrically distributed.
- For a general k, we say that X has a *negative binomial distribution*.
- What is the probability distribution $u(x, k, p)$ of X?

The Poisson Distribution

- The Poisson distribution can be viewed as arising from the binomial distribution, when n is large and p is small.

- The Poisson distribution with parameter λ is obtained as a limit of binomial distributions with parameters n and p, where it was assumed that $np = \lambda$, and $n \to \infty$.

$$P(X = k) \approx \frac{\lambda^k}{k!} e^{-\lambda} \ .$$

Example

- A typesetter makes, on the average, one mistake per 1000 words. Assume that he is setting a book with 100 words to a page.

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- A typesetter makes, on the average, one mistake per 1000 words. Assume that he is setting a book with 100 words to a page.
- Let S_{100} be the number of mistakes that he makes on a single page.
- Then the exact probability distribution for S_{100} would be obtained by considering S_{100} as a result of 100 Bernoulli trials with $p = 1/1000$.

- The Poisson approximation is

$$\frac{e^{-.1}(.1)^j}{j!}.$$

Exercise

The Poisson distribution with parameter $\lambda = .3$ has been assigned for the outcome of an experiment. Let X be the outcome function. Find $P(X = 0)$, $P(X = 1)$, and $P(X > 1)$.

Exercise

In a class of 80 students, the professor calls on 1 student chosen at random for a recitation in each class period. There are 32 class periods in a term.

1. Write a formula for the exact probability that a given student is called upon j times during the term.

2. Write a formula for the Poisson approximation for this probability. Using your formula estimate the probability that a given student is called upon more than twice.

Hypergeometric Distribution

- Suppose that we have a set of N balls, of which k are red and $N - k$ are blue.
- We choose n of these balls, without replacement, and define X to be the number of red balls in our sample.
- The distribution of X is called *the hypergeometric distribution*.
- Note that this distribution depends upon three parameters, namely N, k, and n.

- We will use the notation $h(N, k, n, x)$ to denote $P(X = x)$.

- The distribution function is

$$h(N, k, n, x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \ .$$

Example

A bridge deck has 52 cards with 13 cards in each of four suits: spades, hearts, diamonds, and clubs. A hand of 13 cards is dealt from a shuffled deck. Find the probability that the hand has

1. a distribution of suits 4, 4, 3, 2 (for example, four spades, four hearts, three diamonds, two clubs).

2. a distribution of suits 5, 3, 3, 2.