Continuous Conditional Probability

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Definition

- If X is a continuous random variable with density function $f(x)$, and if E is an event with positive probability, we define a *conditional density* function by the formula

$$f(x|E) \quad = \quad \begin{cases} f(x)/P(E), & \text{if } x \in E \\ 0, & \text{if } x \notin E. \end{cases}$$

- Then for any event F, we have

$$P(F|E) = \int_F f(x|E)\, dx \ .$$

- The expression $P(F|E)$ is called the *conditional probability* of F given E.

Examples

- Let X be the random variable obtained by squaring a real number chosen at random from $[0, 1]$. Suppose that we know that $X \leq 1/2$. What is the probability that $X \leq 1/4$?

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- Recall
$$F_X(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \sqrt{x}, & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } x \geq 1, \end{cases}$$
and
$$f(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1/(2\sqrt{x}), & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x > 1. \end{cases}$$

- In the dart game, suppose we know that the dart lands in the upper half of the target. What is the probability that its distance from the center is less than $1/2$?

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- Experience has shown that X has an exponential density with some parameter λ, which depends upon the size of the lump:

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

- Suppose that when we perform this experiment, we notice that the clock reads r seconds, and is still running.
- What is the probability that there is no emission in a further s seconds?

Independent Events

- If E and F are two events with positive probability in a continuous sample space, then we define E and F to be *independent* if

$$P(E|F) = P(E)$$

and

$$P(F|E) = P(F).$$

Example

- In the dart game let E be the event that the dart lands in the *upper* half of the target $(y \geq 0)$ and F the event that the dart lands in the *right* half of the target $(x \geq 0)$.
- Are E and F independent?

Joint Cumulative Distribution Function

- Let $X_1,\ X_2, \ldots,\ X_n$ be continuous random variables associated with an experiment, and let $\bar{X} = (X_1,\ X_2, \ldots,\ X_n)$. Then the *joint cumulative distribution* function of $\bar{X}$ is defined by

$$F(x_1, x_2, \ldots, x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n)\ .$$

Joint Density Functions

- The joint density function of $\bar{X}$ satisfies the following equation:

$$F(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} f(t_1, t_2, \ldots t_n)\, dt_n dt_{n-1} \ldots dt_1 \,.$$

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- Then

$$f(x_1, x_2, \ldots, x_n) = \frac{\partial^n F(x_1, x_2, \ldots, x_n)}{\partial x_1 \partial x_2 \cdots \partial x_n} \,.$$

Marginal Densities

Theorem. *Let* X *and* Y *be jointly continuous random variables, with joint density function* $f_{X,Y}$*. Then the (marginal) density* f_X *of* X *satisfies*

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy,$$

for all $x \in \mathbb{R}$*. A similar formula holds for* f_Y*.*

Independent Random Variables

- Let X_1, X_2, ..., X_n be continuous random variables with cumulative distribution functions $F_1(x)$, $F_2(x), \ldots,$ $F_n(x)$. Then these random variables are *mutually independent* if

$$F(x_1, x_2, \ldots, x_n) = F_1(x_1)F_2(x_2)\cdots F_n(x_n)$$

for any choice of $x_1, x_2, \ldots, x_n$.

Theorem. *Let* X_1*,* X_2*, . . . ,* X_n *be continuous random variables with density functions* $f_1(x),\ f_2(x), \ldots,\ f_n(x)$*. Then these random variables are* mutually independent *if and only if*

$$f(x_1, x_2, \ldots, x_n) = f_1(x_1)f_2(x_2)\cdots f_n(x_n)$$

for any choice of $x_1, x_2, \ldots, x_n$*.*

Example

Let X and Y be continuous random variables with joint density function $f_{X,Y}$ given by

$$f_{X,Y}(x,y) = \begin{cases} 4x^2y + 2y^5, & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Are X and Y independent?

- Choose a point $\omega = (\omega_1, \omega_2)$ at random from the unit square. Set $X_1 = \omega_1^2$, $X_2 = \omega_2^2$, and $X_3 = \omega_1 + \omega_2$.
  - Are X_1 and X_2 independent?
  - Are X_1 and X_3 independent?